

# NEWTON SERIES AND EXTENDED DERIVATION RELATIONS FOR MULTIPLE $L$ -VALUES

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ABSTRACT. We investigate Newton series for truncated multiple  $L$ -values and thereby obtain a class of relations for multiple  $L$ -values. In addition, we give a formulation and a proof of extended derivation relations for multiple  $L$ -values.

## CONTENTS

1. Introduction	1
2. MPL	3
2.1. Differential Formula	3
2.2. Algebraic Setup	4
2.3. Generalized Landen Connection Formula for MPL's and Inversion Sequences of Truncated MLV's	5
3. Newton Series	8
3.1. Order of the $l$ -th Difference of Truncated MLV's	8
3.2. Basic Properties	12
3.3. Algebraic Preliminary	13
3.4. A Functional Equation	15
3.5. Relations for MLV's	17
4. Extended Derivation	19
4.1. Definitions and Properties	19
4.2. Extended Derivation Relation for MLV's	24
4.3. On the Derivation Relation	28
5. Proofs of Lemmata	30
Appendix: Tables	35
References	36

## 1. INTRODUCTION

Let  $r$  and  $n$  be positive integers. For  $r_1, \dots, r_n \in \mathbb{Z}/r\mathbb{Z}$  and  $k_1, \dots, k_n \in \mathbb{N}$ , two types of multiple  $L$ -values (MLV's), namely, shuffle type ( $\mathfrak{M}$ ) and harmonic type ( $*$ ), were defined in [1] by

$$L^{\mathfrak{M}}(\mathbf{k}; \mathbf{r}) = \lim_{m \rightarrow \infty} \sum_{m > m_1 > \dots > m_n > 0} \frac{\zeta^{r_1(m_1 - m_2)} \dots \zeta^{r_{n-1}(m_{n-1} - m_n)} \zeta^{r_n m_n}}{m_1^{k_1} \dots m_n^{k_n}}$$

and

$$L^*(\mathbf{k}; \mathbf{r}) = \lim_{m \rightarrow \infty} \sum_{m > m_1 > \dots > m_n > 0} \frac{\zeta^{r_1 m_1} \dots \zeta^{r_n m_n}}{m_1^{k_1} \dots m_n^{k_n}},$$

where  $(\mathbf{k}; \mathbf{r})$  denote the index set  $(k_1, \dots, k_n; r_1, \dots, r_n)$  and  $\zeta = \exp(2\pi i/r)$ . We also define two types of non-strict MLV's by

$$\overline{L}^{\mathfrak{m}}(\mathbf{k}; \mathbf{r}) = \lim_{m \rightarrow \infty} \sum_{m \geq m_1 \geq \dots \geq m_n > 0} \frac{\zeta^{r_1(m_1 - m_2)} \dots \zeta^{r_{n-1}(m_{n-1} - m_n)} \zeta^{r_n m_n}}{m_1^{k_1} \dots m_n^{k_n}}$$

and

$$\overline{L}^*(\mathbf{k}; \mathbf{r}) = \lim_{m \rightarrow \infty} \sum_{m \geq m_1 \geq \dots \geq m_n > 0} \frac{\zeta^{r_1 m_1} \dots \zeta^{r_n m_n}}{m_1^{k_1} \dots m_n^{k_n}}.$$

Note that both types of non-strict MLV's converge if  $(k_1, r_1) \neq (1, 0)$  (see [1] for example).

The bound of the dimension of the ‘MLV-space’, which is a  $\mathbb{Q}$ -vector space generated by MLV's, has been studied by P. Deligne and A. Goncharov in [3, 6]. According to their result, there are several linear relations over  $\mathbb{Q}$  among the MLV's. In [1], a large class of relations for MLV's called extended double shuffle relations (EDSR), which contains the derivation relation, was introduced. In the present paper, using the theory of Newton series, we obtain a new class of relations for MLV's containing the extended derivation relation.

Now let  $s_1, \dots, s_n \in \mathbb{C}^\times$ . We define two types of ‘truncated’ MLV's for an index set  $(\mathbf{k}; \mathbf{s}) = (k_1, \dots, k_n; s_1, \dots, s_n)$  by

$$S_{(\mathbf{k}; \mathbf{s})}^{\mathfrak{m}}(m) = \sum_{m \geq m_1 \geq \dots \geq m_n \geq 0} \frac{s_1^{m_1 - m_2} \dots s_{n-1}^{m_{n-1} - m_n} s_n^{m_n + 1}}{(m_1 + 1)^{k_1} \dots (m_n + 1)^{k_n}}$$

and

$$S_{(\mathbf{k}; \mathbf{s})}^*(m) = \sum_{m \geq m_1 \geq \dots \geq m_n \geq 0} \frac{s_1^{m_1 + 1} \dots s_n^{m_n + 1}}{(m_1 + 1)^{k_1} \dots (m_n + 1)^{k_n}},$$

which can be viewed as sequences in terms of  $m \in \mathbb{Z}_{\geq 0}$ . Clearly, non-strict MLV's are limits of these truncated MLV's as  $m \rightarrow \infty$ . More precisely,

$$\overline{L}^{\sharp}(\mathbf{k}; \mathbf{r}) = \lim_{m \rightarrow \infty} S_{(\mathbf{k}; \mathbf{s})}^{\sharp}(m)$$

with  $s_i = \zeta^{r_i}$  ( $i = 1, 2, \dots, n$ ) and  $\sharp = \mathfrak{m}$  (shuffle) or  $*$  (harmonic).

We prove the inversion sequence of a truncated MLV is a linear combination of truncated MLV's in §2. The generalized Landen connection formula of the multiple polylogarithm function plays an important role in the proof, which can be considered as a generalization of the proof of the identity

$$(1) \quad \sum_{i=1}^m (-1)^i \binom{m}{i} \frac{1}{i} = - \sum_{i=1}^m \frac{1}{i}$$

studied by L. Euler in [4].

The inversion sequences of truncated MLV's are required when Newton series of truncated MLV's are discussed. We construct Newton series of truncated MLV's, find several properties of these series, in particular, a functional equation, and obtain a class of relations for MLV's in §3.

In §4, certain extensions of the derivation relation for MLV's are formulated. Several advantageous properties of ‘extended derivation operators’  $\widehat{\partial}_n^{(c)}$  and  $\partial_n^{(c)}$  are introduced and then a proof of the extended derivation relation for MLV's is established.

Proofs of some lemmata are presented in §5. Data of computation using Risa/Asir, an open source general computer algebra system, are given in Appendix at the bottom of the paper.

## 2. MPL

In this section, we prove the generalized Landen connection formula for multiple polylogarithms (MPL's) and find the inversion sequence of a truncated MLV. To describe these or other facts precisely, it is convenient to use the algebraic setup on the non-commutative algebra  $\mathcal{A}_{\mathbb{C}^\times} := \mathbb{Q}\langle x, y_s | s \in \mathbb{C}^\times \rangle$  in infinitely many indeterminates  $x, y_s (s \in \mathbb{C}^\times)$ . We introduce a number of operators on  $\mathcal{A}_{\mathbb{C}^\times}$  to obtain the inversion sequence of a truncated MLV.

**2.1. Differential Formula.** Let  $(\mathbf{k}; \mathbf{s})$  be the index set  $(k_1, \dots, k_n; s_1, \dots, s_n)$  ( $k_i \in \mathbb{N}, s_i \in \mathbb{C}$ ). Here,  $k_1 + \dots + k_n$  is the weight, and  $n$  is the depth.

Next, we introduce the strict and non-strict MPL's (of  $\mathfrak{m}$ -type) for an index set  $(\mathbf{k}; \mathbf{s})$ :

$$\text{Li}_{(\mathbf{k}; \mathbf{s})}^{\mathfrak{m}}(z) = \sum_{m_1 > \dots > m_n > 0} \frac{s_1^{m_1 - m_2} \dots s_{n-1}^{m_{n-1} - m_n} s_n^{m_n}}{m_1^{k_1} \dots m_n^{k_n}} z^{m_1}$$

and

$$\overline{\text{Li}}_{(\mathbf{k}; \mathbf{s})}^{\mathfrak{m}}(z) = \sum_{m_1 \geq \dots \geq m_n > 0} \frac{s_1^{m_1 - m_2} \dots s_{n-1}^{m_{n-1} - m_n} s_n^{m_n}}{m_1^{k_1} \dots m_n^{k_n}} z^{m_1}$$

where  $z$  is a complex variable. These MPL's (with  $s_1 \neq 0$ ) converge absolutely if  $|z| < |1/s_1|$ . For the index set  $(1; s)$  ( $s \in \mathbb{C}$ ), we have

$$\text{Li}_{(1; s)}^{\mathfrak{m}}(z) = -\log(1 - sz)$$

and the identity

$$(2) \quad \text{Li}_{(1; s)}^{\mathfrak{m}}\left(\frac{z}{1 - z}\right) = \text{Li}_{(1; 1-s)}^{\mathfrak{m}}(z) - \text{Li}_{(1; 1)}^{\mathfrak{m}}(z)$$

which are fundamental properties of the logarithm function.

**Lemma 2.1** (Differential formula). *For  $|z| < |1/s_1|$  and an index set  $(\mathbf{k}; \mathbf{s})$  consisting of  $k_1, \dots, k_n \in \mathbb{N}$  and  $s_1, \dots, s_n \in \mathbb{C}^\times$ , we have*

$$\frac{d}{dz} \text{Li}_{(\mathbf{k}; \mathbf{s})}^{\mathfrak{m}}(z) = \begin{cases} \frac{1}{z} \text{Li}_{(k_1-1, k_2, \dots, k_n; \mathbf{s})}^{\mathfrak{m}}(z) & k_1 > 1, \\ \frac{s_1}{1 - s_1 z} \text{Li}_{(k_2, \dots, k_n; s_2, \dots, s_n)}^{\mathfrak{m}}(z) & k_1 = 1, n > 1, \\ \frac{1}{1 - s_1 z} & k_1 = 1, n = 1 \end{cases}$$

and

$$\begin{aligned} & \frac{d}{dz} \overline{\text{Li}}_{(\mathbf{k}; \mathbf{s})}^{\mathfrak{m}}\left(\frac{z}{1 - z}\right) \\ &= \begin{cases} \left(\frac{1}{z} + \frac{1}{1 - z}\right) \overline{\text{Li}}_{(k_1-1, k_2, \dots, k_n; \mathbf{s})}^{\mathfrak{m}}\left(\frac{z}{1 - z}\right) & k_1 > 1, \\ \left(\frac{1 - s_1}{1 - (1 - s_1)z} - \frac{1}{1 - z}\right) \overline{\text{Li}}_{(k_2, \dots, k_n; s_2, \dots, s_n)}^{\mathfrak{m}}\left(\frac{z}{1 - z}\right) & k_1 = 1, n > 1, \\ \frac{1}{1 - (1 - s_1)z} - \frac{1}{1 - z} & k_1 = 1, n = 1. \end{cases} \end{aligned}$$

*Proof.* The proof is simple for  $k_1 > 1$  and for  $k_1 = 1, n = 1$ . If  $k_1 = 1, n > 1$ ,

$$\begin{aligned}
& \frac{d}{dz} \text{Li}_{(1, k_2, \dots, k_n; s_1, \dots, s_n)}^{\mathfrak{m}}(z) \\
&= \sum_{m_1 > \dots > m_n > 0} \frac{s_1^{m_1 - m_2} \dots s_{n-1}^{m_{n-1} - m_n} s_n^{m_n}}{m_2^{k_2} \dots m_n^{k_n}} z^{m_1 - 1} \\
&= \sum_{m_2 > \dots > m_n > 0} \left( \sum_{m_1 = m_2 + 1}^{\infty} s_1^{m_1} z^{m_1 - 1} \right) \frac{s_1^{-m_2} s_2^{m_2 - m_3} \dots s_{n-1}^{m_{n-1} - m_n} s_n^{m_n}}{m_2^{k_2} \dots m_n^{k_n}} \\
&= \sum_{m_2 > \dots > m_n > 0} \frac{s_1^{m_2 + 1} z^{m_2}}{1 - s_1 z} \frac{s_1^{-m_2} s_2^{m_2 - m_3} \dots s_{n-1}^{m_{n-1} - m_n} s_n^{m_n}}{m_2^{k_2} \dots m_n^{k_n}} \\
&= \frac{s_1}{1 - s_1 z} \text{Li}_{(k_2, \dots, k_n; s_2, \dots, s_n)}^{\mathfrak{m}}(z).
\end{aligned}$$

Using  $\frac{d}{dz} \left( \frac{z}{z-1} \right) = -\frac{1}{(z-1)^2}$ , we obtain the second formula.  $\square$

**2.2. Algebraic Setup.** In order to provide precise descriptions, we use an algebraic setup that is similar to that of Arakawa-Kaneko [1]. Let  $\Lambda$  be a group. We denote by  $\mathcal{A}_\Lambda$  the non-commutative algebra  $\mathbb{Q}\langle x, y_s | s \in \Lambda \rangle$  in indeterminates  $x, y_s (s \in \Lambda)$ . The subalgebras  $\mathcal{A}_\Lambda^1$  and  $\mathcal{A}_\Lambda^0$  are given by

$$\mathcal{A}_\Lambda \supset \mathcal{A}_\Lambda^1 := \mathbb{Q} + \sum_{s \in \Lambda} \mathcal{A}_\Lambda y_s \supset \mathcal{A}_\Lambda^0 := \mathbb{Q} + \sum_{s \in \Lambda} x \mathcal{A}_\Lambda y_s + \sum_{s, t \in \Lambda, t \neq 1} y_t \mathcal{A}_\Lambda y_s.$$

We often view an index set  $(\mathbf{k}; \mathbf{s}) = (k_1, \dots, k_n; s_1, \dots, s_n)$  ( $k_i \in \mathbb{N}, s_i \in \Lambda$ ) as a word  $x^{k_1-1} y_{s_1} \dots x^{k_n-1} y_{s_n} \in \mathcal{A}_\Lambda^1$  and vice versa. A word  $w$  is called admissible if  $w \in \mathcal{A}_\Lambda^0$ , and an index set  $(\mathbf{k}; \mathbf{s})$  ( $s \in \Lambda$ ) is called admissible if the corresponding word is admissible. Here, the number of  $x$  and  $y_s$  ( $y_s$ ) of a word  $w$  is its weight (depth), which corresponds with the naming convention of the corresponding index set.

The MPL-evaluation maps  $\text{Li}_\bullet^{\mathfrak{m}}(z), \overline{\text{Li}}_\bullet^{\mathfrak{m}}(z) : \mathcal{A}_\Lambda^1 \rightarrow \mathbb{C}[[z]]$  are defined by  $\mathbb{Q}$ -linearity and

$$\text{Li}_w^{\mathfrak{m}}(z) = \text{Li}_{(\mathbf{k}; \mathbf{s})}^{\mathfrak{m}}(z), \text{Li}_1^{\mathfrak{m}}(z) = 1$$

and

$$\overline{\text{Li}}_w^{\mathfrak{m}}(z) = \overline{\text{Li}}_{(\mathbf{k}; \mathbf{s})}^{\mathfrak{m}}(z), \overline{\text{Li}}_1^{\mathfrak{m}}(z) = 1,$$

respectively, where  $w = x^{k_1-1} y_{s_1} \dots x^{k_n-1} y_{s_n}$ .

As complex-valued sequences from  $\mathbb{Z}_{\geq 0}$  to  $\mathbb{C}$ , two types of truncated MLV's  $\mathbf{S}_{(\mathbf{k}; \mathbf{s})}^{\mathfrak{m}}$  and  $\mathbf{S}_{(\mathbf{k}; \mathbf{s})}^*$  were defined in §1. Here, we also define alternative truncated MLV's  $\mathbf{s}_{(\mathbf{k}; \mathbf{s})}^{\mathfrak{m}}$  and  $\mathbf{s}_{(\mathbf{k}; \mathbf{s})}^*$  for the index set  $(\mathbf{k}; \mathbf{s})$  with  $k_i \in \mathbb{N}$  and  $s_i \in \mathbb{C}^\times$  by

$$\mathbf{s}_{(\mathbf{k}; \mathbf{s})}^{\mathfrak{m}}(m) = \sum_{m = m_1 \geq \dots \geq m_n \geq 0} \frac{s_1^{m_1 - m_2} \dots s_{n-1}^{m_{n-1} - m_n} s_n^{m_n + 1}}{(m_1 + 1)^{k_1} \dots (m_n + 1)^{k_n}}$$

and

$$\mathbf{s}_{(\mathbf{k}; \mathbf{s})}^*(m) = \sum_{m = m_1 \geq \dots \geq m_n \geq 0} \frac{s_1^{m_1 + 1} \dots s_n^{m_n + 1}}{(m_1 + 1)^{k_1} \dots (m_n + 1)^{k_n}}.$$

The evaluation maps of truncated MLV's  $\mathbf{S}_\bullet^\sharp, \mathbf{s}_\bullet^\sharp : \mathcal{A}_{\mathbb{C}^\times}^1 \rightarrow \mathbb{C}$  are also defined by  $\mathbb{Q}$ -linearity and the mappings as well as the MPL-evaluation maps, that is

$$\mathbf{S}_w^\sharp(m) = \mathbf{S}_{(\mathbf{k}; \mathbf{s})}^\sharp(m), \mathbf{S}_1^\sharp(m) = 1$$

and

$$\mathbf{s}_w^\sharp(m) = \mathbf{s}_{(\mathbf{k}; \mathbf{s})}^\sharp(m), \mathbf{s}_1^\sharp(m) = 1$$

where  $w = x^{k_1-1}y_{s_1} \cdots x^{k_n-1}y_{s_n}$  and  $\sharp = \mathfrak{m}$  or  $*$ .

In addition, we introduce three operators on the space of complex-valued sequences  $\mathbb{Z}_{\geq 0} \rightarrow \mathbb{C}$ : the partial-sum operator  $\Sigma$ , its inverse operator  $\Sigma^{-1}$ , and the inversion operator  $\nabla$ , given by

$$(\Sigma a)(m) = \sum_{i=0}^m a(i),$$

$$(\Sigma^{-1}a)(m) = \begin{cases} a(0) & m = 0, \\ a(m) - a(m-1) & m > 0 \end{cases}$$

and

$$(\nabla a)(m) = \sum_{i=0}^m (-1)^i \binom{m}{i} a(i).$$

Note that  $\Sigma\Sigma^{-1} = \Sigma^{-1}\Sigma = \text{id}$ ,  $\nabla^2 = \text{id}$  and  $(\Sigma\nabla)^2 = \text{id}$ .

Under these notations, we find that the identity

$$(3) \quad \mathbf{s}_w^\sharp = \Sigma \mathbf{s}_w^\sharp$$

holds. We study an explicit form of the sequence  $\nabla \mathbf{s}_w^\mathfrak{m}$  in the next subsection.

**2.3. Generalized Landen Connection Formula for MPL's and Inversion Sequences of Truncated MLV's.** Let  $\varphi$  and  $\iota$  be automorphisms on  $\mathcal{A}_{\mathbb{C}^\times}$  given by

$$\varphi(x) = x + y_1, \varphi(y_s) = \delta(s)y_s - y_1$$

and

$$\iota(x) = x, \iota(y_s) = \delta(s)y_{1-s} - (1 - \delta(s))y_1,$$

where  $\delta$  is defined on  $\mathbb{C}$  by

$$\delta(s) = \begin{cases} 0 & s = 0, 1, \\ 1 & \text{otherwise.} \end{cases}$$

Note that  $\varphi^2 = \text{id}$ ,  $\iota^2 = \text{id}$  and  $\varphi\iota = \iota\varphi$ .

**Theorem 2.2** (Generalized Landen Connection Formula). *For any word  $w \in \mathcal{A}_{\mathbb{C}^\times}^1$ , there exists  $\varepsilon > 0$  such that the identity*

$$\text{Li}_w^\mathfrak{m}(z) = \text{Li}_{\varphi\iota(w)}^\mathfrak{m}\left(\frac{z}{z-1}\right)$$

holds in the open disc  $|z| < \varepsilon$ .

*Proof.* Such  $\varepsilon > 0$  can be taken as the minimal number of radii of convergence of appearing MPL's. The identity itself is proven by the induction on the weight of the word  $w$ . If  $w = y_s$ , it is nothing but the identity (7). Since  $\varphi$  and  $\iota$  are automorphisms, we have

$$\begin{aligned} & \text{Li}_{\varphi\iota(x^{k_1-1}y_{s_1} \cdots x^{k_n-1}y_{s_n})}^\mathfrak{m}\left(\frac{z}{z-1}\right) \\ &= \begin{cases} \text{Li}_{(x+y_1)\varphi\iota(x^{k_1-2}y_{s_1}x^{k_2-1}y_{s_2} \cdots x^{k_n-1}y_{s_n})}^\mathfrak{m}\left(\frac{z}{z-1}\right) & k_1 > 1, \\ \text{Li}_{(\delta(s_1)y_{1-s_1}-y_1)\varphi\iota(x^{k_2-1}y_{s_2} \cdots x^{k_n-1}y_{s_n})}^\mathfrak{m}\left(\frac{z}{z-1}\right) & k_1 = 1. \end{cases} \end{aligned}$$

By Lemma 2.1 and the induction hypothesis, we have

$$\begin{aligned}
& \frac{d}{dz} \text{Li}_{\varphi \iota(x^{k_1-1} y_{s_1} \dots x^{k_n-1} y_{s_n})}^{\mathbf{m}} \left( \frac{z}{z-1} \right) \\
&= \begin{cases} \frac{1}{z} \text{Li}_{\varphi \iota(x^{k_1-2} y_{s_1} x^{k_2-1} y_{s_2} \dots x^{k_n-1} y_{s_n})}^{\mathbf{m}} \left( \frac{z}{z-1} \right) & k_1 > 1 \\ \frac{s_1}{1-s_1 z} \text{Li}_{\varphi \iota(x^{k_2-1} y_{s_2} \dots x^{k_n-1} y_{s_n})}^{\mathbf{m}} \left( \frac{z}{z-1} \right) & k_1 = 1, n > 1 \\ \frac{s_1}{1-s_1 z} & k_1 = 1, n = 1 \end{cases} \\
&= \begin{cases} \frac{1}{z} \text{Li}_{x^{k_1-2} y_{s_1} x^{k_2-1} y_{s_2} \dots x^{k_n-1} y_{s_n}}^{\mathbf{m}}(z) & k_1 > 1 \\ \frac{s_1}{1-s_1 z} \text{Li}_{x^{k_2-1} y_{s_2} \dots x^{k_n-1} y_{s_n}}^{\mathbf{m}}(z) & k_1 = 1, n > 1 \\ \frac{s_1}{1-s_1 z} & k_1 = 1, n = 1 \end{cases} \\
&= \frac{d}{dz} \text{Li}_{\varphi \iota(x^{k_1-1} y_{s_1} \dots x^{k_n-1} y_{s_n})}^{\mathbf{m}}(z).
\end{aligned}$$

Integrate both sides from 0 to  $z$ , and we obtain Theorem.  $\square$

Let  $\alpha$  and  $\gamma$  be automorphisms on  $\mathcal{A}_{\mathbb{C}^\times}$  and let  $R_w (w \in \mathcal{A}_{\mathbb{C}^\times})$  be a  $\mathbb{Q}$ -linear operator on  $\mathcal{A}_{\mathbb{C}^\times}$  given by

$$\begin{aligned}
\alpha(x) &= y_1, \alpha(y_s) = (1 - \delta(s))x + \delta(s)y_s, \\
\gamma(x) &= x, \gamma(y_s) = x + y_s
\end{aligned}$$

and

$$R_w(w') = w'w \quad (w' \in \mathcal{A}_{\mathbb{C}^\times}).$$

We define  $\mathbb{Q}$ -linear operators  $\star$  and  $d_{\mathbf{m}}$  on  $\mathcal{A}_{\mathbb{C}^\times}^1$ , respectively, by

$$\star(wy_s) = \alpha \iota(w)(y_1 - \delta(s)y_{1-s})$$

and

$$d_{\mathbf{m}}(wy_s) = \gamma(w)y_s,$$

where  $w \in \mathcal{A}_{\mathbb{C}^\times}$ . Note that  $\alpha^2 = \text{id}$ ,  $\alpha \iota = \iota \alpha$  and  $\star^2 = \text{id}$ . We also find that

$$(4) \quad \overline{\text{Li}}_w^{\mathbf{m}}(z) = \text{Li}_{d_{\mathbf{m}}(w)}^{\mathbf{m}}(z).$$

**Lemma 2.3.**  $\varphi \iota d_{\mathbf{m}} = -d_{\mathbf{m}} \star$ .

*Proof.* For  $w \in \mathcal{A}_{\mathbb{C}^\times}$ , we have

$$\varphi \iota d_{\mathbf{m}}(wy_s) = \varphi \iota(\gamma(w)y_s) = \varphi \iota \gamma(w)(\delta(s)y_{1-s} - y_1)$$

and

$$-d_{\mathbf{m}} \star(wy_s) = -d_{\mathbf{m}}(\alpha(w)(y_1 - \delta(s)y_{1-s})) = \gamma \alpha(w)(\delta(s)y_{1-s} - y_1).$$

It is a simple task to check the identity  $\varphi \iota \gamma = \gamma \alpha$ .  $\square$

Using this lemma, we interpret the generalized Landen connection formula for the strict MPL in Theorem 2.2 as that for the non-strict MPL.

**Corollary 2.4.** For any word  $w \in \mathcal{A}_{\mathbb{C}^\times}^1$ , there exists  $\varepsilon > 0$  such that the identity

$$\overline{\text{Li}}_w^{\mathbf{m}}(z) = -\overline{\text{Li}}_{\star(w)}^{\mathbf{m}}\left(\frac{z}{z-1}\right)$$

holds in the open disc  $|z| < \varepsilon$ .

*Proof.* By equation (4), Theorem 2.2, and Lemma 2.3, we have

$$\text{LHS} = \text{Li}_{d_{\mathbf{m}}(w)}^{\mathbf{m}}(z) = \text{Li}_{\varphi d_{\mathbf{m}}(w)}^{\mathbf{m}}\left(\frac{z}{z-1}\right) = -\text{Li}_{d_{\mathbf{m}}\star(w)}^{\mathbf{m}}\left(\frac{z}{z-1}\right),$$

which is made equivalent to the RHS of the identity by using equation (4) again.  $\square$

**Lemma 2.5.** *There exists  $\varepsilon > 0$  such that the identities*

$$\frac{1}{1-z} \overline{\text{Li}}_w^{\mathbf{m}}(z) = \sum_{m \geq 0} (\Sigma \mathbf{s}_w^{\mathbf{m}})(m) z^{m+1}$$

and

$$\frac{1}{1-z} \overline{\text{Li}}_w^{\mathbf{m}}\left(\frac{z}{z-1}\right) = - \sum_{m \geq 0} (\nabla \Sigma^{-1} \mathbf{s}_w^{\mathbf{m}})(m) z^{m+1}$$

hold in the open disc  $|z| < \varepsilon$ .

*Proof.* Such  $\varepsilon > 0$  can be taken as the minimal number of radii of convergence of appearing MPL's. By the definition, we have

$$\overline{\text{Li}}_w^{\mathbf{m}}(z) = \sum_{m \geq 1} \mathbf{s}_w^{\mathbf{m}}(m-1) z^m.$$

Then,

$$\begin{aligned} \frac{1}{1-z} \overline{\text{Li}}_w^{\mathbf{m}}(z) &= \sum_{l=1}^{\infty} \mathbf{s}_w^{\mathbf{m}}(l-1) \frac{z^l}{1-z} = \sum_{l=1}^{\infty} \mathbf{s}_w^{\mathbf{m}}(l-1) \sum_{m=l}^{\infty} z^m \\ &= \sum_{m > 0} \sum_{l=1}^m \mathbf{s}_w^{\mathbf{m}}(l-1) z^m = \sum_{m \geq 0} (\Sigma \mathbf{s}_w^{\mathbf{m}})(m) z^{m+1} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{1-z} \overline{\text{Li}}_w^{\mathbf{m}}\left(\frac{z}{z-1}\right) &= \sum_{l=1}^{\infty} (-1)^l \mathbf{s}_w^{\mathbf{m}}(l-1) \frac{z^l}{(1-z)^{l+1}} = \sum_{l=1}^{\infty} (-1)^l \mathbf{s}_w^{\mathbf{m}}(l-1) \sum_{m=l}^{\infty} \binom{m}{l} z^m \\ &= \sum_{m > 0} \sum_{l=1}^m (-1)^l \binom{m}{l} \mathbf{s}_w^{\mathbf{m}}(l-1) z^m = - \sum_{m \geq 0} (\nabla \Sigma^{-1} \mathbf{s}_w^{\mathbf{m}})(m) z^{m+1}. \end{aligned}$$

$\square$

Using these properties of MPL's, we obtain the inversion sequences of truncated MLV's.

**Theorem 2.6.** *For any word  $w \in \mathcal{A}_{\mathbb{C}^\times}^{\mathbf{m}}$ , we have*

$$\nabla \mathbf{s}_w^{\mathbf{m}} = \mathbf{s}_{\star(w)}^{\mathbf{m}}.$$

*Proof.* By Corollary 2.4 and Lemma 2.5, we have  $\Sigma \mathbf{s}_w^{\mathbf{m}} = \nabla \Sigma^{-1} \mathbf{s}_{\star(w)}^{\mathbf{m}}$ . Since both  $\nabla$  and  $\Sigma \nabla$  are involutions, we have  $\nabla \mathbf{s}_w^{\mathbf{m}} = \Sigma \nabla \Sigma \mathbf{s}_w^{\mathbf{m}} = \mathbf{s}_{\star(w)}^{\mathbf{m}}$ .  $\square$

**Remark 2.7.** In the case of  $w = y_1$ , we have

$$(5) \quad \sum_{i=0}^m (-1)^i \binom{m}{i} \frac{1}{i+1} = \frac{1}{m+1}$$

for  $m \geq 0$  because  $\star(w) = y_1$  and  $\mathbf{s}_{y_1}^{\mathbf{m}}(m) = \frac{1}{m+1}$ . We can confirm that equation (5) is equivalent to Euler's equation (6).

## 3. NEWTON SERIES

The Newton series for a sequence  $a : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{C}$  is a one-variable complex function that interpolates the sequence  $a$ . In this section, we prove several analytic properties of the Newton series for truncated MLV's and obtain a class of relations for MLV's.

**3.1. Order of the  $l$ -th Difference of Truncated MLV's.** For  $s_1, \dots, s_p \in \mathbb{C}$  and  $m \in \mathbb{Z}_{\geq 0}$ , we set

$$\mathbf{c}_{s_1, \dots, s_p}(m) = \sum_{m=m_1 \geq \dots \geq m_p \geq 0} \frac{s_1^{m_1-m_2} \dots s_{p-1}^{m_{p-1}-m_p}}{(m_1+1) \dots (m_{p-1}+1)} s_p^{m_p}.$$

In addition, for  $s_1, \dots, s_p, t_1, \dots, t_p \in \mathbb{C}$  and  $m, l \in \mathbb{Z}_{\geq 0}$ , we set

$$\begin{aligned} & \mathbf{c}_{s_1, \dots, s_p; t_1, \dots, t_p}(m, l) \\ &= \binom{m+l}{m}^{-1} \sum_{\substack{m=m_1 \geq \dots \geq m_p \geq 0, \\ l=l_1 \geq \dots \geq l_p \geq 0}} \binom{m_1-m_2+l_1-l_2}{m_1-m_2} \dots \binom{m_{p-1}-m_p+l_{p-1}-l_p}{m_{p-1}-m_p} \\ & \quad \times \binom{m_p+l_p}{m_p} \frac{s_1^{m_1-m_2} t_1^{l_1-l_2} \dots s_{p-1}^{m_{p-1}-m_p} t_{p-1}^{l_{p-1}-l_p}}{(m_1+l_1+1) \dots (m_{p-1}+l_{p-1}+1)} s_p^{m_p} t_p^{l_p}. \end{aligned}$$

If  $s_1, \dots, s_p, t_1, \dots, t_p \geq 0$ , we see that  $\mathbf{c}_{s_1, \dots, s_p; t_1, \dots, t_p}(m, l) \geq 0$  for any  $m, l \in \mathbb{Z}_{\geq 0}$ .

**Lemma 3.1.** *Let  $p \geq 2$  and  $s_1, \dots, s_p, t_1, \dots, t_p \in \mathbb{C}$ . For any  $m, l \in \mathbb{Z}_{\geq 0}$ , we have*

$$\begin{aligned} & (m+l+1) \mathbf{c}_{s_1, \dots, s_p; t_1, \dots, t_p}(m, l) - s_1 m \mathbf{c}_{s_1, \dots, s_p; t_1, \dots, t_p}(m-1, l) \\ & - t_1 l \mathbf{c}_{s_1, \dots, s_p; t_1, \dots, t_p}(m, l-1) = \mathbf{c}_{s_2, \dots, s_p; t_2, \dots, t_p}(m, l). \end{aligned}$$

*Proof.* By definition, we have

$$\begin{aligned} & (m+l+1) \mathbf{c}_{s_1, \dots, s_p; t_1, \dots, t_p}(m, l) \\ &= \binom{m+l}{m}^{-1} \left\{ \sum_{\substack{m=m_2 \geq \dots \geq m_p \geq 0, \\ l=l_2 \geq \dots \geq l_p \geq 0}} \binom{m_2-m_3+l_2-l_3}{m_2-m_3} \dots \binom{m_{p-1}-m_p+l_{p-1}-l_p}{m_{p-1}-m_p} \right. \\ & \quad \times \binom{m_p+l_p}{m_p} \frac{s_1^{m_1-m_2} t_1^{l_1-l_2} \dots s_{p-1}^{m_{p-1}-m_p} t_{p-1}^{l_{p-1}-l_p}}{(m_2+l_2+1) \dots (m_{p-1}+l_{p-1}+1)} s_p^{m_p} t_p^{l_p} \\ & + \sum_{\substack{m \geq m_2 \geq \dots \geq m_p \geq 0, \\ l \geq l_2 \geq \dots \geq l_p \geq 0, \\ m > m_2 \text{ or } l > l_2}} \binom{m-m_2+l-l_2}{m-m_2} \binom{m_2-m_3+l_2-l_3}{m_2-m_3} \dots \binom{m_{p-1}-m_p+l_{p-1}-l_p}{m_{p-1}-m_p} \\ & \quad \times \binom{m_p+l_p}{m_p} \frac{s_1^{m_1-m_2} t_1^{l_1-l_2} \dots s_{p-1}^{m_{p-1}-m_p} t_{p-1}^{l_{p-1}-l_p}}{(m_2+l_2+1) \dots (m_{p-1}+l_{p-1}+1)} s_p^{m_p} t_p^{l_p} \Big\}. \end{aligned}$$

Using the identity

$$\binom{m-m_2+l-l_2}{m-m_2} = \binom{m-m_2+l-l_2-1}{m-m_2} + \binom{m-m_2+l-l_2-1}{m-m_2-1},$$

we have

$$\text{RHS} = \mathbf{c}_{s_2, \dots, s_p; t_2, \dots, t_p}(m, l) + \binom{m+l}{m}^{-1} \left\{ \sum_{\substack{m \geq m_2 \geq \dots \geq m_p \geq 0, \\ l > l_2 \geq \dots \geq l_p \geq 0}} \binom{m-m_2+l-l_2-1}{m-m_2} Q \right.$$



$$+ \sum_{\substack{m > m_2 \geq \dots \geq m_p \geq 0, \\ l \geq l_2 \geq \dots \geq l_p \geq 0}} \binom{m - m_2 + l - l_2 - 1}{m - m_2 - 1} Q \Big\}$$

where

$$Q = \binom{m_2 - m_3 + l_2 - l_3}{m_2 - m_3} \dots \binom{m_{p-1} - m_p + l_{p-1} - l_p}{m_{p-1} - m_p} \binom{m_p + l_p}{m_p} \\ \times \frac{t_1^{l-l_2} s_2^{m_2-m_3} t_2^{l_2-l_3} \dots s_{p-1}^{m_{p-1}-m_p} t_{p-1}^{l_{p-1}-l_p}}{(m_2 + l_2 + 1) \dots (m_{p-1} + l_{p-1} + 1)} s_p^{m_p} t_p^{l_p}.$$

This implies the identity of Lemma.  $\square$

We introduce an operator on the space of complex-valued sequences called the difference operator  $\Delta$  given by

$$(\Delta a)(m) = a(m) - a(m+1)$$

for any sequence  $a : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{C}$ .

**Proposition 3.2.** *Let  $s_1, \dots, s_p \in \mathbb{C}$  and  $m, l \in \mathbb{Z}_{\geq 0}$ . We have*

$$(\Delta^l \mathbf{c}_{s_1, \dots, s_p})(m) = \mathbf{c}_{s_1, \dots, s_p; 1-s_1, \dots, 1-s_p}(m, l).$$

*Proof.* Set the generating functions

$$f_{s_1, \dots, s_p}(X, Y) = \sum_{m, l=0}^{\infty} (\Delta^l \mathbf{c}_{s_1, \dots, s_p})(m) \frac{X^m Y^l}{m! l!},$$

$$g_{s_1, \dots, s_p}(X, Y) = \sum_{m, l=0}^{\infty} \mathbf{c}_{s_1, \dots, s_p; 1-s_1, \dots, 1-s_p}(m, l) \frac{X^m Y^l}{m! l!}.$$

Then, we need only show that

$$f_{s_1, \dots, s_p}(X, Y) - g_{s_1, \dots, s_p}(X, Y) = 0,$$

which is equivalent to showing that

$$\begin{cases} \text{i) } f_{s_1, \dots, s_p}(X, 0) - g_{s_1, \dots, s_p}(X, 0) = 0, \\ \text{ii) } (\partial_X + \partial_Y - 1)(f_{s_1, \dots, s_p}(X, Y) - g_{s_1, \dots, s_p}(X, Y)) = 0. \end{cases}$$

Identity i) is trivial. Since  $(\partial_X + \partial_Y - 1)f_{s_1, \dots, s_p}(X, Y) = 0$ , we need only show

$$(6) \quad (\partial_X + \partial_Y - 1)g_{s_1, \dots, s_p}(X, Y) = 0.$$

The proof is by induction on  $p$ . Since  $g_{s_1}(X, Y) = e^{s_1 X + (1-s_1)Y}$ , the identity (6) holds. Suppose that  $p \geq 2$ . By Lemma 3.1, we have

$$(7) \quad (X\partial_X + Y\partial_Y + 1 - s_1 X - (1-s_1)Y)g_{s_1 \dots s_p}(X, Y) = g_{s_2 \dots s_p}(X, Y).$$

Based on the identity (7), the identity

$$[\partial_X + \partial_Y - 1, X\partial_X + Y\partial_Y + 1 - s_1 X - (1-s_1)Y] = \partial_X + \partial_Y - 1,$$

and the induction hypothesis, we have

$$(X\partial_X + Y\partial_Y + 2 - s_1 X - (1-s_1)Y)(\partial_X + \partial_Y - 1)g_{s_1 \dots s_p}(X, Y) \\ = (\partial_X + \partial_Y - 1)(X\partial_X + Y\partial_Y + 1 - s_1 X - (1-s_1)Y)g_{s_1 \dots s_p}(X, Y) = 0.$$

Since the map  $X\partial_X + Y\partial_Y + 2 - s_1 X - (1-s_1)Y : \mathbb{C}[[X, Y]] \rightarrow \mathbb{C}[[X, Y]]$  is injective, we obtain  $(\partial_X + \partial_Y - 1)g_{s_1, \dots, s_p}(X, Y) = 0$ .  $\square$

This proposition implies that, for  $0 \leq s_i \leq 1$  ( $0 \leq i \leq 1$ ), we have

$$(8) \quad (\Delta^l \mathbf{c}_{s_1, \dots, s_p})(m) \geq 0$$

for any  $m, l \geq 0$ .

**Proposition 3.3.** *Let  $l \in \mathbb{Z}_{\geq 0}$ . Suppose that  $|s_i| < 1$  or  $s_i = 1$  for any  $0 \leq i \leq 1$ . For any  $\varepsilon > 0$ , we have*

$$(\Delta^l \mathbf{c}_{s_1, \dots, s_p, 0})(m) = O\left(\frac{1}{m^{l+1-\varepsilon}}\right)$$

as  $m \rightarrow \infty$ .

*Proof.* Let  $t_i = 1 - s_i$  ( $1 \leq i \leq p$ ). By Proposition 3.2,

$$\begin{aligned} & |(\Delta^l \mathbf{c}_{s_1, \dots, s_p, 0})(m)| = |\mathbf{c}_{s_1, \dots, s_p, 0; t_1, \dots, t_p, 1}(m, l)| \\ &= \left| \sum_{\substack{m=m_1 \geq \dots \geq m_p \geq 0, \\ l=l_1 \geq \dots \geq l_{p+1} \geq 0}} \binom{m_1 - m_2 + l_1 - l_2}{m_1 - m_2} \dots \binom{m_{p-1} - m_p + l_{p-1} - l_p}{m_{p-1} - m_p} \binom{m_p + l_p - l_{p+1}}{m_p} \right. \\ & \quad \times \frac{s_1^{m_1 - m_2} t_1^{l_1 - l_2} \dots s_{p-1}^{m_{p-1} - m_p} t_{p-1}^{l_{p-1} - l_p} s_p^{m_p} t_p^{l_p - l_{p+1}}}{(M_1 + \dots + M_p + L_1 + \dots + L_{p+1} + 1) \dots (M_p + L_p + L_{p+1} + 1)} \Big| \\ &\leq \sum_{\substack{m=M_1+\dots+M_p, \\ l=L_1+\dots+L_{p+1}, \\ M_i \geq 0(1 \leq i \leq p), \\ L_j \geq 0(1 \leq j \leq p+1)}} \binom{M_1 + L_1}{M_1} \dots \binom{M_p + L_p}{M_p} \prod_{\nu=1}^p \frac{|s_\nu|^{M_\nu} |t_\nu|^{L_\nu}}{M_\nu + \dots + M_p + L_\nu + \dots + L_{p+1} + 1}. \end{aligned}$$

Set  $I = \{1 \leq i \leq p | s_i = 1\}$  and  $J = \{1 \leq i \leq p | s_i \neq 1\}$ . Then,

$$\begin{aligned} \text{RHS} &= \sum_{\substack{m=M_1+\dots+M_p, \\ l=L_1+\dots+L_{p+1}, \\ M_i \geq 0(1 \leq i \leq p), \\ L_j \geq 0(1 \leq j \leq p+1), \\ i \in I \Rightarrow L_i = 0}} \left\{ \prod_{i \in I} \frac{1}{M_i + \dots + M_p + L_i + \dots + L_{p+1} + 1} \right\} \\ &\quad \times \left\{ \prod_{i \in J} \binom{M_i + L_i}{M_i} \frac{|s_i|^{M_i} |t_i|^{L_i}}{M_i + \dots + M_p + L_i + \dots + L_{p+1} + 1} \right\}. \end{aligned}$$

Since  $|t_i| \leq 1 + |s_i| \leq 2$  ( $1 \leq i \leq p$ ),

$$\begin{aligned} \text{RHS} &\leq \sum_{\substack{m=M_1+\dots+M_p, \\ l=L_1+\dots+L_{p+1}, \\ M_i \geq 0(1 \leq i \leq p), \\ L_j \geq 0(1 \leq j \leq p+1), \\ i \in I \Rightarrow L_i = 0}} \left\{ \prod_{i \in I} \frac{1}{M_i + \dots + M_p + 1} \right\} \left\{ \prod_{i \in J} \binom{M_i + l}{M_i} \frac{|s_i|^{M_i} 2^{L_i}}{M_i + \dots + M_p + 1} \right\} \\ &\leq 2^l \sum_{\substack{m=M_1+\dots+M_p, \\ l=L_1+\dots+L_{p+1}, \\ M_i \geq 0(1 \leq i \leq p), \\ L_j \geq 0(1 \leq j \leq p+1), \\ i \in I \Rightarrow L_i = 0}} \left\{ \prod_{i \in I} \frac{1}{M_i + \dots + M_p + 1} \right\} \left\{ \prod_{i \in J} \binom{M_i + l}{M_i} \frac{|s_i|^{M_i}}{M_i + \dots + M_p + 1} \right\} \end{aligned}$$

$$(9) \leq 2^l(l+1)^{\#J+1} \sum_{\substack{m=M_1+\dots+M_p, \\ M_i \geq 0(1 \leq i \leq p)}} \left\{ \prod_{i \in I} \frac{1}{M_i + \dots + M_p + 1} \right\} \left\{ \prod_{i \in J} \binom{M_i + l}{M_i} \frac{|s_i|^{M_i}}{M_i + \dots + M_p + 1} \right\}.$$

If  $1 \in I$ , then

$$\begin{aligned} (9) &= \frac{2^l(l+1)^{\#J+1}}{m+1} \sum_{\substack{m=M_1+\dots+M_p, \\ M_i \geq 0(1 \leq i \leq p)}} \left\{ \prod_{i \in I \setminus \{1\}} \frac{1}{M_i + \dots + M_p + 1} \right\} \left\{ \prod_{i \in J} \binom{M_i + l}{M_i} \frac{|s_i|^{M_i}}{M_i + \dots + M_p + 1} \right\} \\ &\leq \frac{2^l(l+1)^{\#J+1}}{m+1} \sum_{\substack{m=M_1+\dots+M_p, \\ M_i \geq 0(1 \leq i \leq p)}} \left\{ \prod_{i \in I \setminus \{1\}} \frac{1}{M_i + 1} \right\} \left\{ \prod_{i \in J} \binom{M_i + l}{M_i} |s_i|^{M_i} \right\} \\ &\leq \frac{2^l(l+1)^{\#J+1}}{m+1} \sum_{M_2, \dots, M_p=0}^m \left\{ \prod_{i \in I \setminus \{1\}} \frac{1}{M_i + 1} \right\} \left\{ \prod_{i \in J} \binom{M_i + l}{M_i} |s_i|^{M_i} \right\} \\ (10) &\leq \frac{2^l(l+1)^{\#J+1}}{m+1} \left\{ \prod_{i \in I \setminus \{1\}} \left( \sum_{M_i=0}^m \frac{1}{M_i + 1} \right) \right\} \left\{ \prod_{i \in J} \left( \sum_{M_i=0}^m \binom{M_i + l}{M_i} |s_i|^{M_i} \right) \right\}. \end{aligned}$$

For  $0 \leq x < 1$ , we have

$$\sum_{M=0}^{\infty} \binom{M+l}{M} x^M = \frac{1}{l!} \sum_{M=0}^{\infty} (M+1) \cdots (M+l) x^M < \infty,$$

and hence

$$(10) = O\left(\frac{(\log m)^{\#I-1}}{m}\right)$$

as  $m \rightarrow \infty$ . If  $1 \in J$ , then

$$\begin{aligned} (9) &\leq 2^l(l+1)^{\#J+1} \frac{1}{m+1} \sum_{\substack{m=M_1+\dots+M_p, \\ M_i \geq 0(1 \leq i \leq p)}} \left\{ \prod_{i \in I} \frac{1}{M_i + 1} \right\} \left\{ \prod_{i \in J} \binom{M_i + l}{M_i} |s_i|^{M_i} \right\} \\ &\leq 2^l(l+1)^{\#J+1} \frac{1}{m+1} \sum_{M_1, \dots, M_p=0}^m \left\{ \prod_{i \in I} \frac{1}{M_i + 1} \right\} \left\{ \prod_{i \in J} \binom{M_i + l}{M_i} |s_i|^{M_i} \right\} \\ &\leq 2^l(l+1)^{\#J+1} \frac{1}{m+1} \left\{ \prod_{i \in I} \left( \sum_{M_i=0}^m \frac{1}{M_i + 1} \right) \right\} \left\{ \prod_{i \in J} \left( \sum_{M_i=0}^m \binom{M_i + l}{M_i} |s_i|^{M_i} \right) \right\} \\ &= O\left(\frac{(\log m)^{\#I}}{m}\right) \end{aligned}$$

as  $m \rightarrow \infty$ . Thus, we conclude the Proposition.  $\square$

**Corollary 3.4.** *Let  $l \in \mathbb{Z}_{\geq 0}$ . Suppose that  $|s_i| < 1$  or  $s_i = 1$  for any  $0 \leq i \leq 1$ . Then, for any  $\varepsilon > 0$ , we have*

$$(\Delta^l \mathbf{s}_{(1, \dots, 1; s_1, \dots, s_p)}^{\mathbf{m}})(m) = O\left(\frac{1}{m^{l+1-\varepsilon}}\right)$$

as  $m \rightarrow \infty$ .

*Proof.* The proof follows from Proposition 3.3 and  $\mathbf{s}_{(1,\dots,1;s_1,\dots,s_p)}^{\mathbf{m}} = s_p \mathbf{c}_{s_1,\dots,s_p,0}$ .  $\square$

**3.2. Basic Properties.** Let  $a : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{C}$  be a complex-valued sequence. The Newton series for the sequence  $a$  is defined by

$$f_a(z) := \sum_{n=0}^{\infty} (-1)^n (\nabla a)(n) \binom{z}{n},$$

where  $\binom{z}{n} = \frac{z(z-1)\cdots(z-n+1)}{n!}$  and  $z$  is a complex variable. We find that  $f_a(m) = a(m)$  holds for any  $m \in \mathbb{Z}_{\geq 0}$ . In this sense, we may denote  $f_a(z)$  by  $a(z)$ .

The following properties are basic in the theory of the Newton series (see [5, 9] for details).

**Proposition 3.5.** *Let  $a : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{C}$  be a sequence and  $z \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$ . Then, the series*

$$(11) \quad \sum_{n=0}^{\infty} (-1)^n a(n) \binom{z}{n}$$

*and the Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{z+1}}$$

*possess one and the same abscissa of convergence and absolute convergence.*

**Corollary 3.6.** *If  $a(n) = O(1/n^\varepsilon)$  as  $n \rightarrow \infty$ , the Newton series*

$$\sum_{n=0}^{\infty} (-1)^n a(n) \binom{z}{n}$$

*converges absolutely for  $\operatorname{Re}(z) > -\varepsilon$ .*

**Proposition 3.7.** *Let  $a : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{C}$  be a sequence. If there exists  $N \in \mathbb{Z}_{\geq 0}$  such that  $a(n) = 0$  for  $\mathbb{Z}_{\geq 0} \ni n \geq N$ , then we have  $f_a(z) = 0$  in the entire right half plane of convergence.*

**Lemma 3.8.** *Denote the abscissa of convergence of Newton series*

$$f(z) = \sum_{n=0}^{\infty} (-1)^n a(n) \binom{z}{n}$$

*by  $\rho$ . Let  $l \in \mathbb{Z}_{\geq 0}$ . Then, we have*

$$(-1)^l \binom{z}{l} \sum_{n=0}^{\infty} (-1)^n a(n) \binom{z}{n} = \sum_{n=l}^{\infty} (-1)^n \binom{n}{l} (\Delta^l a)(n-l) \binom{z}{n}$$

*for  $\operatorname{Re}(z) > \rho + l$ .*

**Proposition 3.9.** *Set the Newton series*

$$f(z) = \sum_{n=0}^{\infty} (-1)^n a(n) \binom{z}{n}, g(z) = \sum_{n=0}^{\infty} (-1)^n b(n) \binom{z}{n}.$$

*Denote by  $\rho$  the abscissa of convergence of  $f(z)$ . Let  $\varepsilon > 0$ , and suppose that the sequences  $a, b : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{C}$  satisfy the following conditions:*

- i)  $a(m), (\Delta^l b)(m) \geq 0$  for any  $m, l \in \mathbb{Z}_{\geq 0}$ ,
- ii)  $\rho < 0$ ,
- iii) For any  $l \in \mathbb{Z}_{\geq 0}$ ,  $(\Delta^l b)(m) = O(1/m^{l+\varepsilon})$  as  $m \rightarrow \infty$ .

Then, the product  $f(z)g(z)$  is expressed as a Newton series that converges for  $\operatorname{Re}(z) > \max\{\rho, -\varepsilon\}$ .

*Proof.* Let  $\rho'$  denote the abscissa of convergence of  $g(z)$ . By Lemma 3.8,

$$(12) \quad (-1)^l \binom{z}{l} g(z) = \sum_{n=l}^{\infty} (-1)^n \binom{n}{l} (\Delta^l b)(n-l) \binom{z}{n}$$

for  $\operatorname{Re}(z) > \rho' + l$ . According to iii),

$$\binom{n}{l} (\Delta^l b)(n-l) = O\left(\frac{1}{n^\varepsilon}\right) \quad (n \rightarrow \infty)$$

for any  $l \in \mathbb{Z}_{\geq 0}$ , and hence, by Corollary 3.6, the right-hand side of (12) converges for  $\operatorname{Re}(z) > -\varepsilon$ . Since the left-hand side of (12) also converges for  $\operatorname{Re}(z) > -\varepsilon$ , the identity (12) holds for  $\operatorname{Re}(z) > -\varepsilon$ . Then, we have

$$(13) \quad f(z)g(z) = \sum_{l=0}^{\infty} \sum_{n=l}^{\infty} a(l) (-1)^n \binom{n}{l} (\Delta^l b)(n-l) \binom{z}{n}$$

for  $\operatorname{Re}(z) > \max\{\rho, -\varepsilon\}$ . Suppose that  $z \in \mathbb{R}$  satisfies  $\max\{\rho, -\varepsilon\} < z < 0$ . Then, each term of the right-hand side of (13) is non-negative, and hence we have

$$f(z)g(z) = \sum_{n=0}^{\infty} (-1)^n \left\{ \sum_{l=0}^n a(l) \binom{n}{l} (\Delta^l b)(n-l) \right\} \binom{z}{n}.$$

This shows the Proposition.  $\square$

**3.3. Algebraic Preliminary.** Let  $\Lambda$  be a group, and let  $z_{k,s}$  denote  $x^{k-1}y_s \in \mathcal{A}_\Lambda$ . Note that every word  $w$  with  $\deg(w) > 0$  in  $\mathcal{A}_\Lambda$  can be expressed as  $z_{k_1,s_1} \cdots z_{k_n,s_n} x^l$  for  $k_i \in \mathbb{N}, s_i \in \Lambda (1 \leq i \leq n)$  and some  $l \geq 0$ . Here, we introduce five operators:  $\mathcal{I}, \mathcal{I}^{-1}, M_s (s \in \Lambda), L_w (w \in \mathcal{A}_\Lambda)$ , and  $d_*$ . The operators  $\mathcal{I}, \mathcal{I}^{-1}, M_s (s \in \Lambda)$ , and  $L_w (w \in \mathcal{A}_\Lambda)$  are  $\mathbb{Q}$ -linear maps on  $\mathcal{A}_\Lambda$  defined by

$$\mathcal{I}(z_{k_1,s_1} \cdots z_{k_n,s_n} x^l) = z_{k_1,s_1} z_{k_2,s_1 s_2} \cdots z_{k_n,s_1 \cdots s_n} x^l,$$

$$\mathcal{I}^{-1}(z_{k_1,s_1} \cdots z_{k_n,s_n} x^l) = z_{k_1,s_1} z_{k_2,\frac{s_2}{s_1}} \cdots z_{k_n,\frac{s_n}{s_{n-1}}} x^l,$$

$$M_s(z_{k_1,s_1} \cdots z_{k_n,s_n} x^l) = z_{k_1,ss_1} z_{k_2,s_2} \cdots z_{k_n,s_n} x^l$$

and

$$L_w(w') = ww' \quad (w' \in \mathcal{A}_\Lambda).$$

The operator  $d_*$  is a  $\mathbb{Q}$ -linear map on  $\mathcal{A}_\Lambda^1$  defined by

$$d_* = \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}$$

where  $d_{\mathbf{m}}$  has been given in §2.3.

**Lemma 3.10.** (i)  $\mathcal{I}\mathcal{I}^{-1} = \mathcal{I}^{-1}\mathcal{I} = \operatorname{id}$ .

(ii)  $M_s M_t = M_{st} \quad (s, t \in \Lambda)$ .

(iii)  $\mathcal{I} L_{z_{k,s}} = L_{z_{k,s}} \mathcal{I} M_s$ .

(iv) Each of  $\mathcal{I}, \mathcal{I}^{-1}$  and  $M_s$  commutes with  $L_x$ .

(v)  $d_*$  commutes with  $M_s$ .

*Proof.* The proof of statements (i) through (iv) is simple. Let  $N_s = \mathcal{I}M_s\mathcal{I}^{-1}$ . We see that  $N_s$  is a  $\mathbb{Q}$ -linear map on  $\mathcal{A}_\Lambda$  given by

$$N_s(z_{k_1, s_1} \cdots z_{k_n, s_n} x^l) = z_{k_1, ss_1} \cdots z_{k_n, ss_n} x^l.$$

The operators  $N_s$  and  $d_{\mathfrak{m}}$  commute because

$$\begin{aligned} d_{\mathfrak{m}} N_s(z_{k_1, s_1} \cdots z_{k_n, s_n}) &= d_{\mathfrak{m}}(z_{k_1, ss_1} \cdots z_{k_n, ss_n}) \\ &= x^{k_1-1}(x + y_{ss_1}) \cdots x^{k_n-1}(x + y_{ss_{n-1}}) x^{k_n-1} y_{ss_n} \end{aligned}$$

and

$$\begin{aligned} N_s d_{\mathfrak{m}}(z_{k_1, s_1} \cdots z_{k_n, s_n}) &= N_s(x^{k_1-1}(x + y_{s_1}) \cdots x^{k_n-1}(x + y_{s_{n-1}}) x^{k_n-1} y_{s_n}) \\ &= x^{k_1-1}(x + y_{ss_1}) \cdots x^{k_n-1}(x + y_{ss_{n-1}}) x^{k_n-1} y_{ss_n}. \end{aligned}$$

Then,

$$d_* M_s = \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I} M_s = \mathcal{I}^{-1} d_{\mathfrak{m}} N_s \mathcal{I} = \mathcal{I}^{-1} N_s d_{\mathfrak{m}} \mathcal{I} = M_s \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I} = M_s d_*$$

and we obtain (v).  $\square$

Note that the operator  $\mathcal{I}$  plays a role in translation between  $\mathfrak{m}$ -type and  $*$ -type. For instance, identities

$$(14) \quad \mathbf{S}_w^*(z) = \mathbf{S}_{\mathcal{I}(w)}^{\mathfrak{m}}(z), \quad \mathbf{s}_w^*(z) = \mathbf{s}_{\mathcal{I}(w)}^{\mathfrak{m}}(z)$$

hold.

Next, we define harmonic products.

**Definition 3.11.** Let  $\Lambda$  be a group. We define the harmonic product  $*$  :  $\mathcal{A}_\Lambda^1 \times \mathcal{A}_\Lambda^1 \rightarrow \mathcal{A}_\Lambda^1$  by  $\mathbb{Q}$ -bilinearity and

- (i)  $1 * w = w * 1 = w$ , for any  $w \in \mathcal{A}_\Lambda^1$ ,
- (ii)  $z_{k,s} w * z_{l,t} w' = z_{k,s}(w * z_{l,t} w') + z_{l,t}(z_{k,s} w * w') + z_{k+l,st}(w * w')$   
for any  $k, l \geq 1$  and any words  $w, w' \in \mathcal{A}_\Lambda^1$ .

We define another harmonic product  $\bar{*}$  :  $\mathcal{A}_\Lambda^1 \times \mathcal{A}_\Lambda^1 \rightarrow \mathcal{A}_\Lambda^1$  by  $\mathbb{Q}$ -bilinearity and

- (i)  $1 \bar{*} w = w \bar{*} 1 = w$ , for any  $w \in \mathcal{A}_\Lambda^1$ ,
- (ii)  $z_{k,s} w \bar{*} z_{l,t} w' = z_{k,s}(w \bar{*} z_{l,t} w') + z_{l,t}(z_{k,s} w \bar{*} w') - z_{k+l,st}(w \bar{*} w')$   
for any  $k, l \geq 1$  and any words  $w, w' \in \mathcal{A}_\Lambda^1$ .

In addition, we define two harmonic products  $\dot{*}, \ddot{*}$  :  $\mathcal{A}_\Lambda^1 \times \mathcal{A}_\Lambda^1 \rightarrow \mathcal{A}_\Lambda^1$  by  $\mathbb{Q}$ -bilinearity and

- (i)  $1 \dot{*} w = w \dot{*} 1 = w$ , for any  $w \in \mathcal{A}_\Lambda^1$ ,
- (ii)  $z_{k,s} w \dot{*} z_{l,t} w' = z_{k+l,st}(w * w')$   
for any  $k, l \geq 1$  and any words  $w, w' \in \mathcal{A}_\Lambda^1$ ,

and

- (i)  $1 \ddot{*} w = w \ddot{*} 1 = w$ , for any  $w \in \mathcal{A}_\Lambda^1$ ,
- (ii)  $z_{k,s} w \ddot{*} z_{l,t} w' = z_{k+l,st}(w \bar{*} w')$   
for any  $k, l \geq 1$  and any words  $w, w' \in \mathcal{A}_\Lambda^1$ ,

respectively.

As in [7], all of the four variations of harmonic products are shown to be associative and commutative. Note that the evaluation map  $\mathbf{S}_\bullet^*$  is a  $\bar{*}$ -homomorphism and the evaluation map  $\mathbf{s}_\bullet^*$  is a  $\dot{*}$ -homomorphism, i.e.,

$$(15) \quad \mathbf{S}_w^*(z) \mathbf{S}_{w'}^*(z) = \mathbf{S}_{w \bar{*} w'}^*(z), \quad \mathbf{s}_w^*(z) \mathbf{s}_{w'}^*(z) = \mathbf{s}_{w \dot{*} w'}^*(z).$$

For  $s \in \mathbb{C}^\times$ , let  $F_s$  denote  $M_{\frac{1}{1-\delta(s)s}} \mathcal{I}^{-1} \iota \mathcal{I} M_{(1-\delta(s))+\delta(s)s}$ . The following properties can be shown algebraically by induction. The proofs are a somewhat long and are given in §5.

**Lemma 3.12.** *Let  $\Lambda$  be a group. For any  $w, w' \in \mathcal{A}_\Lambda^1$ , we have*

- (i)  $d_*(w \bar{*} w') = d_*(w) * d_*(w')$ ,
- (i)'  $d_*^{-1}(w * w') = d_*^{-1}(w) \bar{*} d_*^{-1}(w')$ ,
- (ii)  $d_*(w \dot{*} w') = d_*(w) \dot{*} d_*(w')$ ,
- (ii)'  $d_*^{-1}(w \dot{*} w') = d_*^{-1}(w) \dot{*} d_*^{-1}(w')$ .

**Lemma 3.13.** *Let  $\Lambda$  be a subgroup of  $\mathbb{C}^\times$  which contains an element  $1-s$  for any  $s \in \Lambda \setminus \{1\}$ . For any  $w \in \mathcal{A}_\Lambda^1$ , any  $w' \in \mathcal{A}_{\{1\}}^1$ , and any  $s \in \mathbb{C}^\times$ , we have*

$$F_s d_*^{-1}(w) \bar{*} d_*^{-1}(w') = F_s d_*^{-1}(w * w').$$

**3.4. A Functional Equation.** We define the Newton series for the truncated MLV  $\mathbf{s}_w^*(n)$  by

$$\mathbf{s}_w^*(z) = \sum_{n=0}^{\infty} (-1)^n (\nabla \mathbf{s}_w^*)(n) \binom{z}{n}.$$

If the subscript  $w$  is a linear combination of words, the Newton series is also regarded as the corresponding linear combination of Newton series for each appearing word. We prove the following properties of convergence. (We have  $\mathbf{s}_w^*(n) = \mathbf{s}_{\mathcal{I}(w)}^{\mathbf{m}}(n)$ , and hence  $(\nabla \mathbf{s}_w^*)(n) = \mathbf{s}_{*\mathcal{I}(w)}^{\mathbf{m}}(n)$  because of Theorem 2.6.)

**Proposition 3.14.** *Let  $|s_i| \leq 1$  for any  $1 \leq i \leq p$ . Then, the Newton series*

$$(16) \quad \sum_{n=0}^{\infty} (-1)^n \mathbf{s}_{y_{s_1} \dots y_{s_p}}^{\mathbf{m}}(n) \binom{z}{n}$$

*converges absolutely for  $\operatorname{Re}(z) > -1$ . If  $s_1 \neq 1$ , (16) converges for  $\operatorname{Re}(z) > -2$ .*

*Proof.* Since  $|\mathbf{s}_{y_{s_1} \dots y_{s_p}}^{\mathbf{m}}(m)| = O((\log m)^{p-1}/m)$  as  $m \rightarrow \infty$ , we have the first assertion because of Corollary 3.6. Let  $s_1 \neq 1$ . By Proposition 3.5, it is sufficient to show that the Dirichlet series

$$\sum_{m=0}^{\infty} \frac{\mathbf{s}_{y_{s_1} \dots y_{s_p}}^{\mathbf{m}}(m)}{(m+1)^{z+1}}$$

converges for  $\operatorname{Re}(z) > -2$ . Set  $T(M) = \sum_{m=0}^M s_1^m = \frac{1-s_1^{M+1}}{1-s_1}$ . Using the Abel summation method, we have

$$\begin{aligned} \sum_{m=0}^M \frac{\mathbf{s}_{y_{s_1} \dots y_{s_p}}^{\mathbf{m}}(m)}{(m+1)^{z+1}} &= \sum_{M=m_1 \geq \dots \geq m_p \geq 0} \frac{T(m_1 - m_2) s_2^{m_2 - m_3} \dots s_{p-1}^{m_{p-1} - m_p} s_p^{m_p + 1}}{(m_1 + 1)^{z+2} (m_2 + 1) \dots (m_p + 1)} \\ &+ \sum_{M-1 \geq m_1 \geq \dots \geq m_p \geq 0} \frac{T(m_1 - m_2) s_2^{m_2 - m_3} \dots s_{p-1}^{m_{p-1} - m_p} s_p^{m_p + 1}}{(m_2 + 1) \dots (m_p + 1)} \left\{ \frac{1}{(m_1 + 1)^{z+2}} - \frac{1}{(m_1 + 2)^{z+2}} \right\}. \end{aligned}$$

Since

$$\sum_{M \geq m_2 \geq \dots \geq m_p \geq 0} \frac{1}{(m_2 + 1) \dots (m_p + 1)} = O((\log M)^{p-1}) \quad (m \rightarrow \infty),$$

there exists a constant  $C_1$  such that

$$\begin{aligned} & \left| \sum_{M=m_1 \geq \dots \geq m_p \geq 0} \frac{T(m_1 - m_2) s_2^{m_2 - m_3} \dots s_{p-1}^{m_{p-1} - m_p} s_p^{m_p + 1}}{(m_1 + 1)^{z+2} (m_2 + 1) \dots (m_p + 1)} \right| \\ & \leq \frac{C_1}{(M + 1)^{\sigma+2}} \sum_{M \geq m_2 \geq \dots \geq m_p \geq 0} \frac{1}{(m_2 + 1) \dots (m_p + 1)} \rightarrow 0 \quad (M \rightarrow \infty). \end{aligned}$$

On the other hand, there exists a constant  $C_2$  such that

$$\begin{aligned} & \sum_{m_1=0}^{\infty} \left| \sum_{m_1 \geq \dots \geq m_p \geq 0} \frac{T(m_1 - m_2) s_2^{m_2 - m_3} \dots s_{p-1}^{m_{p-1} - m_p} s_p^{m_p + 1}}{(m_2 + 1) \dots (m_p + 1)} \left\{ \frac{1}{(m_1 + 1)^{z+2}} - \frac{1}{(m_1 + 2)^{z+2}} \right\} \right| \\ & \leq C_2 \sum_{m_1=0}^{\infty} \sum_{m_1 \geq \dots \geq m_p \geq 0} \frac{1}{(m_2 + 1) \dots (m_p + 1)} \left| \frac{1}{(m_1 + 1)^{z+2}} - \frac{1}{(m_1 + 2)^{z+2}} \right|. \end{aligned}$$

Set  $\sigma = \operatorname{Re}(z)$ . We find that

$$\left| \frac{1}{(m_1 + 1)^{z+2}} - \frac{1}{(m_1 + 2)^{z+2}} \right| = O\left(\frac{1}{m_1^{\sigma+2}}\right) \quad (m_1 \rightarrow \infty).$$

Hence,

$$\sum_{m_1 \geq \dots \geq m_p \geq 0} \frac{1}{(m_2 + 1) \dots (m_p + 1)} \left| \frac{1}{(m_1 + 1)^{z+2}} - \frac{1}{(m_1 + 2)^{z+2}} \right| = O\left(\frac{(\log m_1)^{p-1}}{m_1^{\sigma+3}}\right)$$

as  $m \rightarrow \infty$ . Therefore, we conclude the Proposition.  $\square$

**Proposition 3.15.** i) Let  $w = x^{k_1-1} y_{s_1} \dots x^{k_n-1} y_{s_n}$ ,  $s_i (1 \leq i \leq n) \in \mathbb{C}^\times$  with  $|1 - s_1 \dots s_i| \leq 1$  for any  $1 \leq i \leq n$ . Then, the Newton series  $\mathbf{s}_w^*(z)$  converges absolutely for  $\operatorname{Re}(z) > -1$ . If  $k_1 = 1$ ,  $\mathbf{s}_w^*(z)$  converges for  $\operatorname{Re}(z) > -2$ .

ii) Let  $w = x^{k_1-1} y_{s_1} \dots x^{k_n-1} y_{s_n}$ ,  $s_i (1 \leq i \leq n) \in \mathbb{C}^\times$  with  $|1 - s_1 \dots s_i| \leq 1$  for any  $1 \leq i \leq n$ . Then, the Newton series  $\mathbf{S}_w^*(z)$  converges absolutely for  $\operatorname{Re}(z) > -1$ . If  $k_1 = 1$ ,  $\mathbf{S}_w^*(z)$  converges for  $\operatorname{Re}(z) > -2$ .

*Proof.* The proof follows from Remark 2.7 2 and Proposition 3.14.  $\square$

**Proposition 3.16.** Let  $\Lambda = (0, 1] \subset \mathbb{R}$ , and  $w = x^{k_1-1} y_{s_1} \dots x^{k_n-1} y_{s_n}$  and  $w' = x^{k'_1-1} y_{s'_1} \dots x^{k'_{n'}-1} y_{s'_{n'}}$  be words in  $\mathcal{A}_\Lambda^1$ . Then, the Newton series  $\mathbf{s}_w^*(z) \mathbf{S}_{w'}^*(z)$  is expressed as a Newton series that converges for  $\operatorname{Re}(z) > -1$ .

*Proof.* The proof follows from Remark 2.7 2, (8), Corollary 3.4, and Proposition 3.9.  $\square$

**Theorem 3.17.** Let  $\Lambda = (0, 1] \subset \mathbb{R}$ , and  $w = x^{k_1-1} y_{s_1} \dots x^{k_n-1} y_{s_n}$  and  $w' = x^{k'_1-1} y_{s'_1} \dots x^{k'_{n'}-1} y_{s'_{n'}}$  be words in  $\mathcal{A}_\Lambda^1$ , and  $s \in \Lambda$ . Then, we have

$$\mathbf{s}_{L_{y_s}(w)}^*(z) \mathbf{S}_{w'}^*(z) = \mathbf{s}_{L_{y_s}(w \boxtimes w')}^*(z)$$

for  $\operatorname{Re}(z) > -2$ .

*Proof.* We find that each term of the right-hand side of the identity converges for  $\operatorname{Re}(z) > -2$ . By Proposition 3.16, the left-hand side is also expressed as a Newton series. Assuming Lemma 3.13, the identity holds on  $\mathbb{Z}_{\geq 0}$ . Therefore, the assertion is proven by Proposition 3.7.  $\square$



**3.5. Relations for MLV's.** For a positive integer  $r$ , we denote the set of  $r$ -th roots of unity by  $\mu_r$ . The MLV's  $L^\sharp$  and  $\overline{L}^\sharp$  ( $\sharp = \mathfrak{m}$  or  $*$ ) were defined in §1. We define the MLV-evaluation maps  $\mathcal{L}^\sharp, \overline{\mathcal{L}}^\sharp : \mathcal{A}_{\mu_r}^0 \rightarrow \mathbb{C}$  by  $\mathbb{Q}$ -linearity and

$$\mathcal{L}^\sharp(x^{k_1-1}y_{s_1} \cdots x^{k_n-1}y_{s_n}) = L^\sharp(k_1, \dots, k_n; r_1, \dots, r_n), \mathcal{L}^\sharp(1) = 1$$

and

$$\overline{\mathcal{L}}^\sharp(x^{k_1-1}y_{s_1} \cdots x^{k_n-1}y_{s_n}) = \overline{L}^\sharp(k_1, \dots, k_n; r_1, \dots, r_n), \overline{\mathcal{L}}^\sharp(1) = 1,$$

respectively, where  $s_i = \zeta^{r_i}$  ( $1 \leq i \leq n$ ) with  $\zeta = \exp(2\pi i/r)$ , which is a primitive  $r$ -th root of unity. We find that equation  $\mathcal{L}^*(w) = \mathcal{L}^\mathfrak{m}(\mathcal{I}(w))$  holds as well as equation (14).

**Proposition 3.18.** *Let  $\Lambda$  be a subgroup of  $\mathbb{C}^\times$  which contains an element  $1-s$  for any  $s \in \Lambda \setminus \{1\}$ . In addition, let  $w$  be a word of  $\mathcal{A}_\Lambda^1$ , and let  $s$  be an element of  $\Lambda$ . For any non-negative integer  $j$ , we have*

$$\frac{1}{j!} \left( \frac{d}{dz} \right)^j \mathbf{s}_{\mathcal{I}^{-1}\iota L_{y_s}(w)}^*(-1) = (-1)^{j+1} \sum_{m=0}^{\infty} \mathbf{s}_{L_x^{-1}d_*^{-1}(\mathcal{I}^{-1}L_{x+\delta(s)y_s}\varphi d_\mathfrak{m}(w) \dot{*} y_1^{j+1})}^*(m).$$

In particular, if  $\Lambda = \mu_r$  for a positive integer  $r$ , we have

$$\frac{1}{j!} \left( \frac{d}{dz} \right)^j \mathbf{s}_{\mathcal{I}^{-1}\iota L_{y_s}d_\mathfrak{m}^{-1}\mathcal{I}M_s(w)}^*(-1) = (-1)^{j+1} \mathcal{L}^\mathfrak{m}(L_x^{-1}\mathcal{I}(\mathcal{I}^{-1}L_{x+\delta(s)y_s}\varphi \mathcal{I}M_s(w) \dot{*} y_1^{j+1})).$$

*Proof.* Generally,  $j$ -th coefficients of Taylor expansion at  $z = -1$  of Newton series  $a(z) = \sum_{n=0}^{\infty} (-1)^j (\nabla a)(n) \binom{z}{n}$  can be expressed as

$$\frac{1}{j!} a^{(j)}(-1) = (-1)^j \sum_{n=0}^{\infty} (\nabla a)(n) \mathbf{s}_{d_*^{-1}(y_1^j)}^*(n-1)$$

(see [11, Lemma 4.6]). Using the identities

$$\mathbf{s}_{\mathcal{I}^{-1}(w)}^* = \mathbf{s}_w^\mathfrak{m}(z)$$

and

$$\mathbf{s}_{d_*^{-1}(y_1^j)}^*(n-1) = (n+1) \mathbf{s}_{d_*^{-1}(y_1^{j+1})}^*(n)$$

and Theorem 2.6, we have

$$\begin{aligned} & \frac{1}{j!} \left( \frac{d}{dz} \right)^j \mathbf{s}_{\mathcal{I}^{-1}\iota L_{y_s}(w)}^*(-1) = \frac{1}{j!} \left( \frac{d}{dz} \right)^j \mathbf{s}_{\iota L_{y_s}(w)}^\mathfrak{m}(-1) \\ (17) \quad & = (-1)^j \sum_{n=0}^{\infty} \mathbf{s}_{\star \iota L_{y_s}(w)}^\mathfrak{m}(n) \cdot (n+1) \mathbf{s}_{d_*^{-1}(y_1^{j+1})}^*(n). \end{aligned}$$

Note that  $\star \iota = -d_\mathfrak{m}^{-1}\varphi d_\mathfrak{m}$  because of Lemma 2.3 and  $d_\mathfrak{m}\iota = \iota d_\mathfrak{m}$ . Using

$$\mathbf{s}_{\star \iota L_{y_s}(w)}^\mathfrak{m}(n) = \mathbf{s}_{\mathcal{I}^{-1}\star \iota L_{y_s}(w)}^*(n) = -\mathbf{s}_{\mathcal{I}^{-1}d_\mathfrak{m}^{-1}\varphi d_\mathfrak{m} L_{y_s}(w)}^*(n)$$

and equation (15), we obtain

$$(17) = (-1)^{j+1} \sum_{n=0}^{\infty} (n+1) \mathbf{s}_{\mathcal{I}^{-1}d_\mathfrak{m}^{-1}\varphi d_\mathfrak{m} L_{y_s}(w) \dot{*} d_\mathfrak{m}^{-1}(y_1^{j+1})}^*(n).$$

Since  $\mathcal{I}^{-1}d_\mathfrak{m}^{-1}\varphi d_\mathfrak{m} L_{y_s}(w) \dot{*} d_\mathfrak{m}^{-1}(y_1^{j+1}) \in \mathcal{A}_\Lambda^0$ , we have

$$(n+1) \mathbf{s}_{\mathcal{I}^{-1}d_\mathfrak{m}^{-1}\varphi d_\mathfrak{m} L_{y_s}(w) \dot{*} d_\mathfrak{m}^{-1}(y_1^{j+1})}^*(n) = \mathbf{s}_{L_x^{-1}(\mathcal{I}^{-1}d_\mathfrak{m}^{-1}\varphi d_\mathfrak{m} L_{y_s}(w) \dot{*} d_\mathfrak{m}^{-1}(y_1^{j+1}))}^*(n).$$

By this identity,  $\mathcal{I}^{-1}d_{\mathbf{m}}^{-1} = d_{*}^{-1}\mathcal{I}^{-1}$ , Lemma 3.12 (i)', and  $\varphi d_{\mathbf{m}} L_{y_s} = L_{x+\delta(s)y_s} \varphi d_{\mathbf{m}}$ , we have the first assertion

$$(18) = (-1)^{j+1} \sum_{m=0}^{\infty} \mathbf{s}_{L_x^{-1}d_{*}^{-1}(\mathcal{I}^{-1}L_{x+\delta(s)y_s}\varphi d_{\mathbf{m}}(w) \dot{*} y_1^{j+1})}^{*}(m).$$

Substituting  $d_{\mathbf{m}}^{-1}\mathcal{I}M_s(w)$  into  $w$  in the above equality, we have

$$(19) \quad \frac{1}{j!} \left( \frac{d}{dz} \right)^j \mathbf{s}_{\mathcal{I}^{-1}L_{y_s}d_{\mathbf{m}}^{-1}\mathcal{I}M_s(w)}^{*}(-1) = (-1)^{j+1} \sum_{m=0}^{\infty} \mathbf{s}_{L_x^{-1}d_{*}^{-1}(\mathcal{I}^{-1}L_{x+\delta(s)y_s}\varphi \mathcal{I}M_s(w) \dot{*} y_1^{j+1})}^{*}(m).$$

If  $\Lambda = \mu_r$ , every term of  $L_x^{-1}d_{*}^{-1}(\mathcal{I}^{-1}L_{x+\delta(s)y_s}\varphi \mathcal{I}M_s(w) \dot{*} y_1^{j+1})$  is admissible and converges to a linear combination of MLV's, i.e.,

$$\begin{aligned} (19) &= (-1)^{j+1} \overline{\mathcal{L}}^{*}(L_x^{-1}d_{*}^{-1}(\mathcal{I}^{-1}L_{x+\delta(s)y_s}\varphi \mathcal{I}M_s(w) \dot{*} y_1^{j+1})) \\ &= (-1)^{j+1} \mathcal{L}^{\mathbf{m}}(L_x^{-1}\mathcal{I}(\mathcal{I}^{-1}L_{x+\delta(s)y_s}\varphi \mathcal{I}M_s(w) \dot{*} y_1^{j+1})). \end{aligned}$$

□

**Remark 3.19.** Let  $w \in \mathcal{A}_{\{1\}}^1$ . Differentiating both sides of the identity

$$\mathbf{s}_w^{*}(z) = (z+1)\mathbf{s}_{L_{y_1}(w)}^{*}(z),$$

we see that

$$\frac{1}{j!} \left( \frac{d}{dz} \right)^j \mathbf{s}_{d_{*}^{-1}(w)}^{*}(-1) = \begin{cases} 0 & j = 0, \\ (-1)^j \mathcal{L}^{\mathbf{m}}(L_x^{-1}(L_x \varphi(w') \dot{*} y_1^j)) & j > 0. \end{cases}$$

Let  $\mathcal{A}_{\Lambda, >0}^1$  denote the set of polynomials in  $\mathcal{A}_{\Lambda}^1$  without constant terms.

**Theorem 3.20.** For any  $w \in \mathcal{A}_{\mu_r, >0}^1$ , any  $w' \in \mathcal{A}_{\{1\}, >0}^1$ , and any  $m \geq 0$ , we have

$$\begin{aligned} \sum_{k+l=m, k \geq 0, l > 0} \mathcal{L}^{\mathbf{m}}(L_x^{-1}\mathcal{I}(\mathcal{I}^{-1}L_{x+\delta(s)y_s}\varphi \mathcal{I}M_s(w) \dot{*} y_1^{k+1})) \mathcal{L}^{\mathbf{m}}(L_x^{-1}(L_x \varphi(w') \dot{*} y_1^l)) \\ = \mathcal{L}^{\mathbf{m}}(L_x^{-1}\mathcal{I}(\mathcal{I}^{-1}L_{x+\delta(s)y_s}\varphi \mathcal{I}M_s(w * w') \dot{*} y_1^{m+1})). \end{aligned}$$

If  $m = 0$ , we consider the left-hand side as 0.

*Proof.* First, we find that

$$\begin{aligned} (20) \quad \sum_{\substack{k+l=m, \\ k \geq 0, l > 0}} \text{Li}_{L_x^{-1}\mathcal{I}(\mathcal{I}^{-1}L_{x+\delta(s)y_s}\varphi \mathcal{I}M_s(w) \dot{*} y_1^{k+1})}^{\mathbf{m}}(1) \text{Li}_{L_x^{-1}(L_x \varphi(w') \dot{*} y_1^l)}^{\mathbf{m}}(1) \\ = \text{Li}_{L_x^{-1}\mathcal{I}(\mathcal{I}^{-1}L_{x+\delta(s)y_s}\varphi \mathcal{I}M_s(w * w') \dot{*} y_1^{m+1})}^{\mathbf{m}}(1) \end{aligned}$$

for any  $w \in \mathcal{A}_{(0,1], >0}^1$ , any  $w' \in \mathcal{A}_{\{1\}, >0}^1$ , and any  $m \geq 0$ . Since each term in (20) is holomorphic for  $|s_i| < 1$  ( $1 \leq i \leq n$ ) because of the Identity Theorem. Moreover, we see that each term in (20) is continuous for  $|s_i| \leq 1$  ( $1 \leq i \leq n$ ), hence the identity (20) is valid for  $|s_i| \leq 1$  ( $1 \leq i \leq n$ ). If  $s_i \in \mu_r$ , each value of non-strict MPL's at  $z = 1$  appearing in (20) is nothing but the MLV. Replacing  $w$  with  $d_{\mathbf{m}-1}\mathcal{I}M_s(w)$ , Theorem is proven. □

Finally, we give the case of  $m = 0$ , the linear part of Theorem 3.20, which is shown to contain the extended derivation relation in the next section.

**Corollary 3.21.** Let  $r$  be a positive integer. For any  $s \in \mu_r$ , we have

$$L_{x+\delta(s)y_s}\varphi \mathcal{I}M_s(\mathcal{A}_{\mu_r, >0}^1 * \mathcal{A}_{\{1\}, >0}^1) \subset \text{Ker} \mathcal{L}^{\mathbf{m}}.$$

## 4. EXTENDED DERIVATION

A generalization of the derivation relation was first mentioned in [8] for the multiple zeta value (MZV) case, formulated as a conjecture in [10], and proven in [12] by reducing the relation to the relations found in [11]. In this section, we present a generalization of the derivation relation for the MLV case, which has been suggested to the authors by Masanobu Kaneko. It is worthwhile to verify several advantageous properties of the extended derivation operators  $\widehat{\partial}_n^{(c)}$  and  $\partial_n^{(c)}$ .

Throughout this section, we fix a positive integer  $r$ . As defined in the previous section, let  $\mu_r = \{1, \zeta, \zeta^2, \dots, \zeta^{r-1}\}$  with  $\zeta = \exp(2\pi i/r)$ . In addition, we denote  $x + y_1 (\in \mathcal{A}_{\mu_r})$  by  $z$  in this section.

**4.1. Definitions and Properties.** For  $n \geq 1$ , the derivation operator  $\partial_n : \mathcal{A}_{\mu_r} \rightarrow \mathcal{A}_{\mu_r}$  appeared in [1] satisfies

$$\partial_n = \frac{1}{(n-1)!} \text{ad}(\theta)^{n-1}(\partial_1),$$

where  $\theta$  is a derivation on  $\mathcal{A}_{\mu_r}$  defined by  $\theta(u) = \frac{1}{2}(uz + zu)$  for  $u = x$  or  $y_s$  ( $s \in \mu_r$ ),  $\partial_1 : \mathcal{A}_{\mu_r} \rightarrow \mathcal{A}_{\mu_r}$  a derivation characterized by

$$\partial_1(x) = xy_1, \quad \partial_1(y_s) = -xy_s + y_sy_1 - y_sy_s,$$

and  $\text{ad}(\theta)(\partial) = [\theta, \partial] := \theta\partial - \partial\theta$ . Then, we give an extension of  $\partial_n$  as follows.

**Definition 4.1.** Let  $c$  be a rational number, and let  $H$  be the derivation on  $\mathcal{A}_{\mu_r}$  defined by  $H(w) = \deg(w)w$  for any words  $w \in \mathcal{A}_{\mu_r}$ . For each integer  $n \geq 1$ , we define a  $\mathbb{Q}$ -linear map  $\widehat{\partial}_n^{(c)}$  from  $\mathcal{A}_{\mu_r}$  to  $\mathcal{A}_{\mu_r}$  by

$$\widehat{\partial}_n^{(c)} = \frac{1}{(n-1)!} \text{ad}(\widehat{\theta}^{(c)})^{n-1}(\partial_1),$$

where  $\widehat{\theta}^{(c)}$  is the  $\mathbb{Q}$ -linear map defined by  $\widehat{\theta}^{(c)}(u) = \theta(u)$  ( $u = x$  or  $y_s$ ) and the rule

$$\widehat{\theta}^{(c)}(ww') = \widehat{\theta}^{(c)}(w)w' + w\widehat{\theta}^{(c)}(w') + cH(w)\partial_1(w')$$

for any  $w, w' \in \mathcal{A}_{\mu_r}$ .

If  $c = 0$ , the extended derivation  $\widehat{\partial}_n^{(c)}$  is reduced to the ordinary derivation  $\partial_n$ . It is known in [1] that  $\partial_n(\mathcal{A}_{\mu_r}^0) \subset \text{Ker } \mathcal{L}^{\text{m}}$  holds for any  $n \geq 1$ , which is called the derivation relation. Although, if  $c \neq 0$  and  $n \geq 2$ , the operator  $\widehat{\partial}_n^{(c)}$  is no longer a derivation, and we cannot prove (but can expect experimentally) that the extended derivation relation is contained in the EDSR, we find in Theorem 4.14 or (25) that the extended derivation relation is proven by using Corollary 3.21.

Now, we define the operator  $\widehat{\psi}_n^{(c)}(u)$ , which is closely related to the operator  $\widehat{\partial}_n^{(c)}$ , and then show some properties of this operator.

**Definition 4.2.** Let  $u \in \mathbb{Q} \cdot x + \sum_{s \in \mu_r} \mathbb{Q} \cdot y_s$ . For  $n \geq 1$  and  $c \in \mathbb{Q}$ , we define the sequence of operators  $\{\widehat{\psi}_n^{(c)}(u)\}_{n=1}^{\infty}$  by  $\widehat{\psi}_1^{(c)}(u) = L_{\partial_1(u)}$  and

$$\widehat{\psi}_n^{(c)}(u) = \frac{1}{n-1}([\widehat{\theta}^{(c)}, \widehat{\psi}_{n-1}^{(c)}(u)] - \frac{1}{2}(L_z \widehat{\psi}_{n-1}^{(c)}(u) + \widehat{\psi}_{n-1}^{(c)}(u)L_z) - c\widehat{\psi}_{n-1}^{(c)}(u)\partial_1)$$

for  $n \geq 2$  (where  $L_w$  has been defined in §3.3 by  $L_w(w') = ww'$ ).

Let  $\nu$  be a  $\mathbb{Q}$ -linear operator on  $\mathbb{Q} \cdot x + \sum_{s \in \mu_r} \mathbb{Q} \cdot y_s$  given by  $\nu(x) = 0$  and  $\nu(y_s) = 1$  for any  $s \in \mu_r$ .

**Lemma 4.3.** *Let  $n \geq 1$  and  $c \in \mathbb{Q}$ . There is another sequence of operators  $\{\widehat{\phi}_n^{(c)}\}_{n=0}^\infty$  such that*

$$\widehat{\psi}_n^{(c)}(u) = (-1)^{\nu(u)} (L_x \widehat{\phi}_{n-1}^{(c)} L_{y_1 + \nu(u)(u-y_1)} + \nu(u) L_u \widehat{\phi}_{n-1}^{(c)} L_{u-y_1})$$

where  $u \in \mathbb{Q} \cdot x + \sum_{s \in \mu_r} \mathbb{Q} \cdot y_s$ .

The proof of Lemma 4.3 is given in §.5.

**Corollary 4.4.** *For  $n \geq 1$  and  $c \in \mathbb{Q}$ , we have*

- (i)  $\widehat{\psi}_n^{(c)}(x + y_1) = 0$ .
- (ii)  $\widehat{\psi}_n^{(c)}(x + \delta(s)y_s) = L_{x+\delta(s)y_s} \widehat{\phi}_{n-1}^{(c)} L_{y_1-\delta(s)y_s} \quad (s \in \mu_r)$ .

*Proof.* By Lemma 4.3, we see that

$$\widehat{\psi}_n^{(c)}(x) = L_x \widehat{\phi}_n^{(c)} L_{y_1}, \quad \widehat{\psi}_n^{(c)}(y_1) = -L_x \widehat{\phi}_n^{(c)} L_{y_1},$$

which implies (i) and (ii) for  $s = 1$ . If  $s \neq 1$ , then

$$\widehat{\psi}_n^{(c)}(y_s) = -L_x \widehat{\phi}_n^{(c)} L_{y_s} + L_{y_s} \widehat{\phi}_n^{(c)} L_{y_s-y_1}$$

by Lemma 4.3 and we obtain (ii) for  $s \neq 1$ .  $\square$

Next, we present the commutativity of  $\widehat{\partial}_n^{(c)}$ .

**Proposition 4.5.** *For any  $n, m \geq 1$  and any  $c, c' \in \mathbb{Q}$ , we have  $[\widehat{\partial}_n^{(c)}, \widehat{\partial}_m^{(c')}] = 0$ .*

*Proof.* Let  $n \geq 1$  and  $u \in \mathbb{Q} \cdot x + \sum_{s \in \mu_r} \mathbb{Q} \cdot y_s$ . We prove the following statements  $(A_n)$  and  $(B_n)$  inductively as  $(A_1), (B_1) \Rightarrow (A_2) \Rightarrow (B_2) \Rightarrow (A_3) \Rightarrow \dots$ .

- $(A_n)$   $[\widehat{\partial}_n^{(c)}, L_u] = \widehat{\psi}_n^{(c)}(u)$ .
- $(B_n)$   $[\widehat{\partial}_n^{(c)}, \widehat{\partial}_i^{(c')}] = 0$ , for any  $1 \leq i \leq n$ , and any  $c, c' \in \mathbb{Q}$ .

If  $(B_n)$  is proven, we obtain the assertion.

Here, we present three remarks. First, if  $(A_n)$  is proven, we see

$$(\alpha_n) \quad [\widehat{\partial}_n^{(c)}, L_z] = 0$$

because of Corollary 4.4(i).

Second, let

$$(B_{n,i}) \quad [\widehat{\partial}_n^{(c)}, \widehat{\partial}_i^{(c')}] = 0 \text{ for a fixed } 1 \leq i \leq n \text{ and any } c, c' \in \mathbb{Q}$$

for  $1 \leq i \leq n$ . Showing statement  $(B_n)$  is equivalent to showing statement  $(B_{n,i})$  for all  $1 \leq i \leq n$ . Moreover, setting

$$(B'_{n,i}) \quad [[\widehat{\partial}_n^{(c)}, \widehat{\partial}_i^{(c')}] , L_u] = 0 \text{ for a fixed } 1 \leq i \leq n \text{ and any } c, c' \in \mathbb{Q},$$

we see that each  $(B_{n,i})$  is equivalent to  $(B'_{n,i})$ . Instead of  $(B_n)$ , we show  $(B'_{n,i})$  by induction on  $i$ .

Third, when the statement  $(A_i)$  (hence  $(\alpha_i)$ ) and the statement  $(B_i)$  are proven for any  $1 \leq i \leq n$ , we can consider the commutative polynomial ring  $\mathbb{Q}[L_z, \widehat{\partial}_1^{(c)}, \dots, \widehat{\partial}_n^{(c)}]$ . Define the degree of generators by  $\deg(L_z) = 1$  and  $\deg(\widehat{\partial}_d^{(c)}) = d$ . Let  $\mathbb{Q}[L_z, \widehat{\partial}_1^{(c)}, \dots, \widehat{\partial}_n^{(c)}]_{(d)}$  denote the degree- $d$  homogenous part. Then, we obtain,

$$(\beta_n) \quad \widehat{\phi}_n^{(c)} \in \mathbb{Q}[L_z, \widehat{\partial}_1^{(c)}, \dots, \widehat{\partial}_n^{(c)}]_{(n)}$$

because of the recursive rule (29) of the operator  $\widehat{\phi}_n^{(c)}$ .

We now begin to prove  $(A_n)$  and  $(B_n)$ . Since  $\widehat{\partial}_1^{(c)} = \partial_1$  for any  $c \in \mathbb{Q}$  and

$$[\partial_1, L_u](w) = \partial_1(uw) - u\partial_1(w) = \partial_1(u)w = L_{\partial_1(u)}(w)$$

for  $w \in \mathcal{A}_{\mu_r}$ , the statement  $(A_1)$  holds. The statement  $(B_1)$  is trivial because  $\widehat{\partial}_1^{(c)} = \partial_1^{(c)}$  for any  $c \in \mathbb{Q}$ .

Suppose  $(A_n)$  (hence also  $(\alpha_n)$ ) and  $(B_n)$  are proven. Then,

$$\begin{aligned} n[\widehat{\partial}_{n+1}^{(c)}, L_u] &= [[\widehat{\theta}^{(c)}, \widehat{\partial}_n^{(c)}], L_u] \\ &= -[[\widehat{\partial}_n^{(c)}, L_u], \widehat{\theta}^{(c)}] - [[L_u, \widehat{\theta}^{(c)}], \widehat{\partial}_n^{(c)}] \\ &= [\widehat{\theta}^{(c)}, \widehat{\psi}_n^{(c)}(u)] + [L_{\widehat{\theta}^{(c)}(u)} + cL_u\partial_1, \widehat{\partial}_n^{(c)}] \\ &= [\widehat{\theta}^{(c)}, \widehat{\psi}_n^{(c)}(u)] - [\widehat{\partial}_n^{(c)}, \frac{1}{2}(L_uL_z + L_zL_u) + cL_u\partial_1] \\ &= [\widehat{\theta}^{(c)}, \widehat{\psi}_n^{(c)}(u)] - \frac{1}{2}(L_z\widehat{\psi}_n^{(c)}(u) + \widehat{\psi}_n^{(c)}(u)L_z) - c\widehat{\psi}_n^{(c)}(u)\partial_1. \end{aligned}$$

Hence, we obtain  $[\widehat{\partial}_{n+1}^{(c)}, L_u] = \widehat{\psi}_{n+1}^{(c)}(u)$ , and  $(A_{n+1})$  (hence  $(\alpha_{n+1})$ ) holds.

Next, suppose that all of  $(A_j)$ 's (hence  $(\alpha_j)$ 's) ( $1 \leq j \leq n+1$ ) and all of  $(B_j)$ 's (hence  $(\beta_j)$ 's) ( $1 \leq j \leq n$ ) are proven. We show  $(B'_{n+1,i})$  ( $1 \leq j \leq n+1$ ) by induction on  $i$ .

$$\begin{aligned} &[[\widehat{\partial}_{n+1}^{(c')}, \widehat{\partial}_i^{(c')}], L_u] \\ &= -[[\widehat{\partial}_i^{(c')}, L_u], \widehat{\partial}_{n+1}^{(c')}] - [[L_u, \widehat{\partial}_{n+1}^{(c')}], \widehat{\partial}_i^{(c')}] \\ &= [\widehat{\partial}_{n+1}^{(c')}, \widehat{\psi}_i^{(c')}(u)] - [\widehat{\partial}_i^{(c')}, \widehat{\psi}_{n+1}^{(c')}(u)] \\ &= (-1)^{\nu(u)}([\widehat{\partial}_{n+1}^{(c')}, L_x\widehat{\phi}_{i-1}^{(c')}L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_u\widehat{\phi}_{i-1}^{(c')}L_{u-y_1}] \\ &\quad - [\widehat{\partial}_i^{(c')}, L_x\widehat{\phi}_n^{(c')}L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_u\widehat{\phi}_n^{(c')}L_{u-y_1}]). \end{aligned}$$

We must show that this becomes 0. Here, set

$$\begin{aligned} P &= [\widehat{\partial}_{n+1}^{(c')}, L_x\widehat{\phi}_{i-1}^{(c')}L_{y_1+\nu(u)(u-y_1)}] - [\widehat{\partial}_i^{(c')}, L_x\widehat{\phi}_n^{(c')}L_{y_1+\nu(u)(u-y_1)}], \\ Q &= [\widehat{\partial}_{n+1}^{(c')}, L_u\widehat{\phi}_{i-1}^{(c')}L_{u-y_1}] - [\widehat{\partial}_i^{(c')}, L_u\widehat{\phi}_n^{(c')}L_{u-y_1}]. \end{aligned}$$

Then, we need only show that

$$(21) \quad \nu(u)Q = -P.$$

Note that  $(-1)^{\nu(u)}\nu(u) = -\nu(u)$ .

Suppose  $i = 1$ . Since  $\widehat{\phi}_{i-1}^{(c')} = \text{id}_{\mathcal{A}_{\mu_r}}$ ,  $\widehat{\partial}_i^{(c')} = \partial_1$ , we have

$$[\widehat{\partial}_{n+1}^{(c')}, \widehat{\phi}_{i-1}^{(c')}] = 0, \quad [\widehat{\partial}_i^{(c')}, \widehat{\phi}_n^{(c')}] = 0.$$

Hence,

$$\begin{aligned} P &= \widehat{\psi}_{n+1}^{(c)}(x)L_{y_1+\nu(u)(u-y_1)} + L_x\widehat{\psi}_{n+1}^{(c)}(y_1 + \nu(u)(u-y_1)) \\ &\quad - \widehat{\psi}_i^{(c')}(x)\widehat{\phi}_n^{(c)}L_{y_1+\nu(u)(u-y_1)} - L_x\widehat{\phi}_n^{(c)}\widehat{\psi}_1^{(c')}(y_1 + \nu(u)(u-y_1)) \\ &= L_x\widehat{\phi}_n^{(c)}L_{y_1}L_{y_1+\nu(u)(u-y_1)} - L_xL_x\widehat{\phi}_n^{(c)}L_{y_1} + \nu(u)L_xL_x\widehat{\phi}_n^{(c)}L_{y_1} \\ &\quad + (-1)^{\nu(u)}\nu(u)L_x(L_x\widehat{\phi}_n^{(c)}L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_u\widehat{\phi}_n^{(c)}L_{u-y_1}) \\ &\quad - L_xL_{y_1}\widehat{\phi}_n^{(c)}L_{y_1+\nu(u)(u-y_1)} + L_x\widehat{\phi}_n^{(c)}L_xL_{y_1} - \nu(u)L_x\widehat{\phi}_n^{(c)}L_xL_{y_1} \\ &\quad - (-1)^{\nu(u)}\nu(u)L_x\widehat{\phi}_n^{(c)}(L_xL_{y_1+\nu(u)(u-y_1)} + \nu(u)L_uL_{u-y_1}) \\ &= L_x[\widehat{\phi}_n^{(c)}, L_{y_1}]L_{y_1+\nu(u)(u-y_1)} + L_x[\widehat{\phi}_n^{(c)}, L_x]L_{y_1} - \nu(u)L_x[\widehat{\phi}_n^{(c)}, L_x]L_{y_1} \\ &\quad + \nu(u)L_x[\widehat{\phi}_n^{(c)}, L_x]L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_x[\widehat{\phi}_n^{(c)}, L_u]L_{u-y_1} \\ &= \nu(u)L_x[\widehat{\phi}_n^{(c)}, L_u]L_{u-y_1}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
Q &= \widehat{\psi}_{n+1}^{(c)}(u)L_{u-y_1} + L_u\widehat{\psi}_{n+1}^{(c)}(u-y_1) - \widehat{\psi}_1^{(c')}(u)\widehat{\phi}_n^{(c)}L_{u-y_1} - L_u\widehat{\phi}_n^{(c)}\widehat{\psi}_1^{(c')}(u-y_1) \\
&= (-1)^{\nu(u)}(L_x\widehat{\phi}_n^{(c)}L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_u\widehat{\phi}_n^{(c)}L_{u-y_1})L_{u-y_1} \\
&\quad + (-1)^{\nu(u)}L_u(L_x\widehat{\phi}_n^{(c)}L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_u\widehat{\phi}_n^{(c)}L_{u-y_1}) + L_uL_x\widehat{\phi}_n^{(c)}L_{y_1} \\
&\quad - (-1)^{\nu(u)}(L_xL_{y_1+\nu(u)(u-y_1)} + \nu(u)L_uL_{u-y_1})\widehat{\phi}_n^{(c)}L_{u-y_1} \\
&\quad - (-1)^{\nu(u)}L_u\widehat{\phi}_n^{(c)}(L_xL_{y_1+\nu(u)(u-y_1)} + \nu(u)L_uL_{u-y_1}) - L_u\widehat{\phi}_n^{(c)}L_xL_{y_1} \\
&= (-1)^{\nu(u)}L_x[\widehat{\phi}_n^{(c)}, L_{y_1+\nu(u)(u-y_1)}]L_{u-y_1} - \nu(u)L_u[\widehat{\phi}_n^{(c)}, L_{u-y_1}]L_{u-y_1} \\
&\quad - (-1)^{\nu(u)}L_u[\widehat{\phi}_n^{(c)}, L_x]L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_u[\widehat{\phi}_n^{(c)}, L_u]L_{u-y_1} - L_u[\widehat{\phi}_n^{(c)}, L_x]L_{y_1}.
\end{aligned}$$

Thus, we obtain  $\nu(u)Q = -\nu(u)L_x[\widehat{\phi}_n^{(c)}, L_u]L_{u-y_1} = -P$  and (21) is proven.

Suppose  $i > 1$ . By the induction hypothesis, again,

$$[\widehat{\partial}_{n+1}^{(c)}, \widehat{\phi}_{i-1}^{(c')}] = 0, \quad [\widehat{\partial}_i^{(c')}, \widehat{\phi}_n^{(c)}] = 0,$$

and we have

$$\begin{aligned}
P &= \widehat{\psi}_{n+1}^{(c)}(x)\widehat{\phi}_{i-1}^{(c')}L_{y_1+\nu(u)(u-y_1)} + L_x\widehat{\phi}_{i-1}^{(c')}\widehat{\psi}_{n+1}^{(c)}(y_1 + \nu(u)(u-y_1)) \\
&\quad - \widehat{\psi}_i^{(c')}(x)\widehat{\phi}_n^{(c)}L_{y_1+\nu(u)(u-y_1)} - L_x\widehat{\phi}_n^{(c)}\widehat{\psi}_i^{(c')}(y_1 + \nu(u)(u-y_1)) \\
&= L_x\widehat{\phi}_n^{(c)}L_{y_1}\widehat{\phi}_{i-1}^{(c')}L_{y_1+\nu(u)(u-y_1)} - L_x\widehat{\phi}_{i-1}^{(c')}L_x\widehat{\phi}_n^{(c)}L_{y_1} + \nu(u)L_x\widehat{\phi}_{i-1}^{(c')}L_x\widehat{\phi}_n^{(c)}L_{y_1} \\
&\quad + (-1)^{\nu(u)}\nu(u)L_x\widehat{\phi}_{i-1}^{(c')}(L_x\widehat{\phi}_n^{(c)}L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_u\widehat{\phi}_n^{(c)}L_{u-y_1}) \\
&\quad - L_x\widehat{\phi}_{i-1}^{(c')}L_{y_1}\widehat{\phi}_n^{(c)}L_{y_1+\nu(u)(u-y_1)} + L_x\widehat{\phi}_n^{(c)}L_x\widehat{\phi}_{i-1}^{(c')}L_{y_1} - \nu(u)L_x\widehat{\phi}_n^{(c)}L_x\widehat{\phi}_{i-1}^{(c')}L_{y_1} \\
&\quad - (-1)^{\nu(u)}\nu(u)L_x\widehat{\phi}_n^{(c)}(L_x\widehat{\phi}_{i-1}^{(c')}L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_u\widehat{\phi}_{i-1}^{(c')}L_{u-y_1}) \\
&= L_x([\widehat{\phi}_n^{(c)}, L_{y_1}]\phi_{i-1}^{(c')} - [\phi_{i-1}^{(c')}, L_{y_1}]\widehat{\phi}_n^{(c)})L_{y_1+\nu(u)(u-y_1)} \\
&\quad + L_x([\widehat{\phi}_n^{(c)}, L_x]\phi_{i-1}^{(c')} - [\phi_{i-1}^{(c')}, L_x]\widehat{\phi}_n^{(c)})L_{y_1} \\
&\quad - \nu(u)L_x([\widehat{\phi}_n^{(c)}, L_x]\phi_{i-1}^{(c')} - [\phi_{i-1}^{(c')}, L_x]\widehat{\phi}_n^{(c)})L_{y_1} \\
&\quad + \nu(u)L_x([\widehat{\phi}_n^{(c)}, L_x]\phi_{i-1}^{(c')} - [\phi_{i-1}^{(c')}, L_x]\widehat{\phi}_n^{(c)})L_{y_1+\nu(u)(u-y_1)} \\
&\quad + \nu(u)L_x([\widehat{\phi}_n^{(c)}, L_u]\phi_{i-1}^{(c')} - [\phi_{i-1}^{(c')}, L_u]\widehat{\phi}_n^{(c)})L_{u-y_1} \\
&= \nu(u)L_x([\widehat{\phi}_n^{(c)}, L_u]\phi_{i-1}^{(c')} - [\phi_{i-1}^{(c')}, L_u]\widehat{\phi}_n^{(c)})L_{u-y_1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
 Q &= \widehat{\psi}_{n+1}^{(c)}(u) \widehat{\phi}_{i-1}^{(c')} L_{u-y_1} + L_u \widehat{\phi}_{i-1}^{(c')} \widehat{\psi}_{n+1}^{(c)}(u-y_1) \\
 &\quad - \widehat{\psi}_i^{(c')} (u) \widehat{\phi}_n^{(c)} L_{u-y_1} - L_u \widehat{\phi}_n^{(c)} \widehat{\psi}_i^{(c')} (u-y_1) \\
 &= (-1)^{\nu(u)} (L_x \widehat{\phi}_n^{(c)} L_{y_1+\nu(u)(u-y_1)} + \nu(u) L_u \widehat{\phi}_n^{(c)} L_{u-y_1}) \widehat{\phi}_{i-1}^{(c')} L_{u-y_1} \\
 &\quad + (-1)^{\nu(u)} L_u \widehat{\phi}_{i-1}^{(c')} (L_x \widehat{\phi}_n^{(c)} L_{y_1+\nu(u)(u-y_1)} + \nu(u) L_u \widehat{\phi}_n^{(c)} L_{u-y_1}) \\
 &\quad + L_u \widehat{\phi}_{i-1}^{(c')} L_x \widehat{\phi}_n^{(c)} L_{y_1} \\
 &\quad - (-1)^{\nu(u)} (L_x \widehat{\phi}_{i-1}^{(c')} L_{y_1+\nu(u)(u-y_1)} + \nu(u) L_u \widehat{\phi}_{i-1}^{(c')} L_{u-y_1}) \widehat{\phi}_n^{(c)} L_{u-y_1} \\
 &\quad - (-1)^{\nu(u)} L_u \widehat{\phi}_n^{(c)} (L_x \widehat{\phi}_{i-1}^{(c')} L_{y_1+\nu(u)(u-y_1)} + \nu(u) L_u \widehat{\phi}_{i-1}^{(c')} L_{u-y_1}) \\
 &\quad - L_u \widehat{\phi}_n^{(c)} L_x \widehat{\phi}_{i-1}^{(c')} L_{y_1} \\
 &= (-1)^{\nu(u)} L_x ([\widehat{\phi}_n^{(c)}, L_{y_1+\nu(u)(u-y_1)}] \widehat{\phi}_{i-1}^{(c')} - [\widehat{\phi}_{i-1}^{(c')}, L_{y_1+\nu(u)(u-y_1)}] \widehat{\phi}_n^{(c)}) L_{u-y_1} \\
 &\quad - \nu(u) L_u ([\widehat{\phi}_n^{(c)}, L_{u-y_1}] \widehat{\phi}_{i-1}^{(c')} - [\widehat{\phi}_{i-1}^{(c')}, L_{u-y_1}] \widehat{\phi}_n^{(c)}) L_{u-y_1} \\
 &\quad + (-1)^{\nu(u)} L_u ([\widehat{\phi}_n^{(c)}, L_x] \widehat{\phi}_{i-1}^{(c')} - [\widehat{\phi}_{i-1}^{(c')}, L_x] \widehat{\phi}_n^{(c)}) L_{y_1+\nu(u)(u-y_1)} \\
 &\quad - \nu(u) L_u ([\widehat{\phi}_n^{(c)}, L_u] \widehat{\phi}_{i-1}^{(c')} - [\widehat{\phi}_{i-1}^{(c')}, L_u] \widehat{\phi}_n^{(c)}) L_{u-y_1} \\
 &\quad + L_u ([\widehat{\phi}_n^{(c)}, L_x] \widehat{\phi}_{i-1}^{(c')} - [\widehat{\phi}_{i-1}^{(c')}, L_x] \widehat{\phi}_n^{(c)}) L_{y_1}.
 \end{aligned}$$

Thus, we obtain  $\nu(u)Q = -\nu(u)L_x([\widehat{\phi}_n^{(c)}, L_u]\widehat{\phi}_{i-1}^{(c')} - [\widehat{\phi}_{i-1}^{(c')}, L_u]\widehat{\phi}_n^{(c)})L_{u-y_1} = -P$ , and (21) is proven.  $\square$

As an application, we obtain the following property.

**Proposition 4.6.** *We have*

$$\widehat{\partial}_n^{(c)} L_{x+\delta(s)y_s} (\mathcal{A}_{\mu_r}^1) \subset L_{x+\delta(s)y_s} (\mathcal{A}_{\mu_r}^1).$$

for any integer  $n \geq 1$ , any  $c \in \mathbb{Q}$ , and any  $s \in \mu_r$ .

*Proof.* The proof is given by induction on  $n$ . Based on the definition of  $\widehat{\partial}_1^{(c)} = \partial_1$ , the proposition holds for  $n = 1$ . Suppose the assertion is true until  $n - 1$ . We prove the case of  $n$  by induction on the degrees of words.

Based on (A<sub>n</sub>) in the proof of Proposition 4.5, for  $u \in \mathbb{Q} \cdot x + \sum_{s \in \mu_r} \mathbb{Q} \cdot y_s$ , we have

$$\begin{aligned}
 \widehat{\partial}_n^{(c)}(u) &= (-1)^{\nu(u)} (L_x \widehat{\phi}_{n-1}^{(c)}(y_1 + \nu(u)(u - y_1)) + \nu(u) L_u \widehat{\phi}_{n-1}^{(c)}(u - y_1)) \\
 &= \begin{cases} L_x \widehat{\phi}_{n-1}^{(c)}(y_1) & u = x, \\ -L_x \widehat{\phi}_{n-1}^{(c)}(y_1) & u = y_1, \\ -L_x \widehat{\phi}_{n-1}^{(c)}(y_s) - L_{y_s} \widehat{\phi}_{n-1}^{(c)}(y_s - y_1) & u = y_s, s \neq 1. \end{cases}
 \end{aligned}$$

By ( $\beta_n$ ) in the proof of Proposition 4.5 and the induction hypothesis, we have  $\widehat{\partial}_n^{(c)}(u) \in L_{x+\delta(s)y_s} (\mathcal{A}_{\mu_r}^1)$ , and the assertion is proven for degree-1 words.

Suppose this is proven until degree  $< d$  and let  $\deg(uw) = d$  with  $u \in \mathbb{Q} \cdot x + \sum_{s \in \mu_r} \mathbb{Q} \cdot y_s$ . Based on  $(A_n)$ , we have

$$\begin{aligned} \widehat{\partial}_n^{(c)}(uw) &= L_u \widehat{\partial}_n^{(c)}(w) + (-1)^{\nu(u)} (L_x \widehat{\phi}_{n-1}^{(c)} L_{y_1 + \nu(u)(u-y_1)} + \nu(u) L_u \widehat{\phi}_{n-1}^{(c)} L_{u-y_1})(w) \\ &= \begin{cases} L_x \widehat{\partial}_n^{(c)}(w) + L_x \widehat{\phi}_{n-1}^{(c)} L_{y_1}(w) & u = x, \\ L_{y_1} \widehat{\partial}_n^{(c)}(w) - L_x \widehat{\phi}_{n-1}^{(c)} L_{y_1}(w) & u = y_1, \\ L_{y_s} \widehat{\partial}_n^{(c)}(w) - L_x \widehat{\phi}_{n-1}^{(c)} L_{y_s}(w) - L_{y_s} \widehat{\phi}_{n-1}^{(c)} L_{y_s-y_1}(w) & u = y_s, s \neq 1. \end{cases} \end{aligned}$$

Accordingly, we see

$$\widehat{\partial}_n^{(c)}(xw) \in x\mathcal{A}_{\mu_r}, \widehat{\partial}_n^{(c)}(y_s w) \in x\mathcal{A}_{\mu_r} + y_s\mathcal{A}_{\mu_r} \ (s \neq 1).$$

Moreover, if  $uw = w'y_s$  ( $s \in \mu_r$ ), we obtain

$$\widehat{\partial}_n^{(c)}(w'y_s) \in \sum_{s \in \mu_r} \mathcal{A}_{\mu_r} y_s$$

based on  $(\beta_n)$  and the induction hypothesis. Combining these statements, we obtain the assertion for degree- $d$  words.  $\square$

**4.2. Extended Derivation Relation for MLV's.** In this subsection, we show the extended derivation relation for MLV's by reducing the relation to Corollary 3.21.

Denote by  $\mathcal{A}_{\mu_r, n}^1$  the weight- $n$  homogenous part of  $\mathcal{A}_{\mu_r}^1$ . Recall  $z_{k,s} = x^{k-1}y_s$  for  $k \geq 1$  and  $s \in \mu_r$  as defined in §3.3. Let  $\mathfrak{W}$  be the  $\mathbb{Q}$ -vector space generated by  $\{\mathcal{H}_w | w \in \mathcal{A}_{\{1\}}^1\}$ , where  $\mathcal{H}_w$  denotes the  $\mathbb{Q}$ -linear operator given by  $\mathcal{H}_w(w') = w * w'$ , and let  $\mathfrak{W}_n$  be the vector subspace of  $\mathfrak{W}$  generated by  $\{\mathcal{H}_w | w \in \mathcal{A}_{\{1\}, n}^1\}$ . Let  $\mathfrak{W}'$  be the  $\mathbb{Q}$ -vector space generated by  $\{L_{z_{k,1}} \mathcal{H}_w | 1 \leq k, w \in \mathcal{A}_{\{1\}}^1\}$ , and let  $\mathfrak{W}'_n$  be the vector subspace of  $\mathfrak{W}'$  generated by  $\{L_{z_{k,1}} \mathcal{H}_w | 1 \leq k \leq n, w \in \mathcal{A}_{\{1\}, n-k}^1\}$ . We define a  $\mathbb{Q}$ -linear map  $\lambda : \mathfrak{W}' \rightarrow \mathfrak{W}$  by  $\lambda(L_{z_{k,1}} \mathcal{H}_w) = \mathcal{H}_{z_{k,1}} w$ .

**Remark 4.7.** The map  $\lambda$  is well-defined for the reasons described in the case of MZV's (see [12]).

**Lemma 4.8.** *For any  $X \in \mathfrak{W}'$ , any  $k \geq 1$ , and any  $s \in \mu_r$ , we have*

$$[\lambda(X), L_{z_{k,s}}] = X L_{z_{k,s}} + M_s L_{x^k} X.$$

*Proof.* It is sufficient to show the case of  $X = L_{z_{k,1}} \mathcal{H}_w$ , but it follows straightforward from the harmonic product rule:

$$(22) \quad [\mathcal{H}_{z_{k,1}w}, L_{z_{l,s}}] = L_{z_{k,1}} \mathcal{H}_w L_{z_{l,s}} + M_s L_{x^l} L_{z_{k,1}} \mathcal{H}_w \ (s \in \mu_r).$$

$\square$

**Lemma 4.9.** *For any  $k, l \geq 1$  and any  $s \in \mu_r$ , we have*

$$(\lambda - 1)(\mathfrak{W}'_k) L_{z_{l,s}} \subset M_s \cdot \mathfrak{W}'_{k+l}.$$

*Proof.* This lemma is given by interpreting (22) as

$$\mathcal{H}_{z_{k,1}w} L_{z_{l,s}} - L_{z_{k,1}} \mathcal{H}_w L_{z_{l,s}} = M_s (L_{z_{l,1}} \mathcal{H}_{z_{k,1}w} + L_{x^l} L_{z_{k,1}} \mathcal{H}_w).$$

$\square$

**Lemma 4.10.** *For any  $k, l \geq 1$  and any  $s \in \mu_r$ , we have*

$$(\lambda - 1)(\mathfrak{W}'_k) \cdot (\lambda - 1)(\mathfrak{W}'_l) \subset (\lambda - 1)(\mathfrak{W}'_{k+l}).$$



*Proof.* Let  $d$  and  $d'$  be the weights of words  $w$  and  $w'$ , respectively. We need only show that  $(\lambda - 1)(L_{z_k} \mathcal{H}_w) \cdot (\lambda - 1)(L_{z_l} \mathcal{H}_{w'}) \in (\lambda - 1)(\mathfrak{W}'_{k+l+d+d'})$ .

$$\begin{aligned}
\text{LHS} &= (\mathcal{H}_{z_k,1} w - L_{z_k,1} \mathcal{H}_w)(\mathcal{H}_{z_l,1} w' - L_{z_l,1} \mathcal{H}_{w'}) \\
&= \mathcal{H}_{z_k,1} w * z_{l,1} w' - \mathcal{H}_{z_k,1} w L_{z_l,1} \mathcal{H}_{w'} - L_{z_k,1} \mathcal{H}_{w * z_{l,1} w'} + L_{z_k,1} \mathcal{H}_w L_{z_l,1} \mathcal{H}_{w'} \\
&= \mathcal{H}_{z_k,1} (w * z_{l,1} w') + z_{l,1} (z_{k,1} w * w') + z_{k+l,1} (w * w') - (L_{z_k,1} \mathcal{H}_w L_{z_l,1} \\
&\quad + L_{z_l,1} \mathcal{H}_{z_k,1} w + L_{z_{k+l,1}} \mathcal{H}_w) \mathcal{H}_{w'} - L_{z_k,1} \mathcal{H}_{w * z_{l,1} w'} + L_{z_k,1} \mathcal{H}_w L_{z_l,1} \mathcal{H}_{w'} \\
&= \mathcal{H}_{z_k,1} (w * z_{l,1} w') - L_{z_k,1} \mathcal{H}_{w * z_{l,1} w'} + \mathcal{H}_{z_{l,1} (z_{k,1} w * w')} - L_{z_{l,1}} \mathcal{H}_{z_k,1} w * w' \\
&\quad + \mathcal{H}_{z_{k+l,1}} (w * w') - L_{z_{k+l,1}} \mathcal{H}_{w * w'} \\
&= (\lambda - 1)(L_{z_k,1} \mathcal{H}_{w * z_{l,1} w'} + L_{z_{l,1}} \mathcal{H}_{z_k,1} w * w' + L_{z_{k+l,1}} \mathcal{H}_{w * w'}). \\
&\in \text{RHS}.
\end{aligned}$$

Hence, the lemma is proven.  $\square$

**Lemma 4.11.** *For any  $X \in \mathfrak{W}'$ , we have  $\lambda(X)(1) = X(1)$ .*

*Proof.*

$$(\lambda - 1)(L_{z_k,1} \mathcal{H}_w)(1) = \mathcal{H}_{z_k,1} w(1) - L_{z_k,1} \mathcal{H}_w(1) = z_{k,1} w - z_{k,1} w = 0.$$

$\square$

**Lemma 4.12.** *Let  $X \in \mathfrak{W}$ . If  $X(1) = 0$  and  $[X, L_{z_{k,s}}] = 0$  for any  $k \geq 1$  and any  $s \in \mu_r$ , we have  $X = 0$  on  $\mathcal{A}_{\mu_r}^1$ .*

*Proof.* If  $[X, L_{z_{k,s}}] = 0$  for any  $k \geq 1$  and any  $s \in \mu_r$ ,

$$X(z_{k_1, s_1} \cdots z_{k_n, s_n}) = z_{k_1, s_1} X(z_{k_2, s_2} \cdots z_{k_n, s_n}) = \cdots = z_{k_1, s_1} \cdots z_{k_n, s_n} X(1) = 0.$$

$\square$

Now, let  $\sigma_s$  ( $s \in \mu_r$ ) denote  $\varphi \mathcal{I} M_s$ . The following proposition is the key to connect the extended derivation operator  $\widehat{\partial}_n^{(c)}$  with the harmonic product operator  $\mathcal{H}_w$ .

**Proposition 4.13.** *Let  $n \geq 1$ ,  $c \in \mathbb{Q}$  and  $s \in \mu_r$ . Then, the following two statements,  $(C_n)$  and  $(D_n)$  hold.*

$$\begin{aligned}
(C_n) \quad & \sigma_s^{-1} \widehat{\phi}_{n-1}^{(c)} L_{y_1 - \delta(s)y_s} \sigma_s \in \mathfrak{W}'_n, \\
(D_n) \quad & \sigma_s^{-1} L_{x + \delta(s)y_s}^{-1} \widehat{\partial}_n^{(c)} L_{x + \delta(s)y_s} \sigma_s = \lambda (\sigma_s^{-1} \widehat{\phi}_{n-1}^{(c)} L_{y_1 - \delta(s)y_s} \sigma_s) \in \mathfrak{W}_n \text{ on } \mathcal{A}_{\mu_r}^1.
\end{aligned}$$

Note that, by Proposition 4.6, the expression  $L_{x + \delta(s)y_s}^{-1}$  in  $(D_n)$  has the well-defined meaning.

*Proof.* We prove inductively that  $(C_1) \Rightarrow (D_1) \Rightarrow (C_2) \Rightarrow (D_2) \Rightarrow (C_3) \Rightarrow \cdots$ . Since

$$\sigma_s^{-1} \widehat{\phi}_0^{(c)} L_{y_1 - \delta(s)y_s} \sigma_s = -\sigma_s^{-1} \varphi L_{y_s} \mathcal{I} M_s = -\sigma_s^{-1} \varphi \mathcal{I} M_s L_{y_1} = -L_{y_1},$$

the assertion  $(C_1)$  holds. Suppose that the above statement is true until  $(C_n)$ . Note that

$$(23) \quad L_{x + \delta(s)y_s}^{-1} \widehat{\partial}_n^{(c)} L_{x + \delta(s)y_s} = \widehat{\partial}_n^{(c)} + \widehat{\phi}_{n-1}^{(c)} L_{y_1 - \delta(s)y_s}$$

based on  $(A_n)$  in the proof of Proposition 4.5 and Corollary 4.4 (ii). In addition, we note that  $(C_n)$  implies the operator  $\sigma_s^{-1} \widehat{\phi}_{n-1}^{(c)} L_{y_1 - \delta(s)y_s} \sigma_s$  is independent of  $s \in \mu_r$ .

First, we prove that the operator  $\sigma_s^{-1}\widehat{\partial}_n^{(c)}\sigma_s$  is also independent of  $s \in \mu_r$  by induction on the  $y_s$ -degree of a word. If  $w = x^l$  ( $l \geq 0$ ), we have

$$\sigma_s^{-1}\widehat{\partial}_n^{(c)}\sigma_s(w) = \sigma_s^{-1}\widehat{\partial}_n^{(c)}(z^l) = 0$$

and hence the claim holds. Suppose that  $\sigma_s^{-1}\widehat{\partial}_n^{(c)}\sigma_s(w')$  ( $w'$  is a word of  $\mathcal{A}_{\mu_r}$ ) that is independent of  $s \in \mu_r$ , and let  $w = z_{l,t}w'$  ( $l \geq 1, t \in \mu_r$ ). We find that

$$\begin{aligned} \sigma(w) &= \varphi \mathcal{I} M_s L_{z_{l,t}}(w') \\ &= \varphi L_{z_{l,t}} \mathcal{I} M_{st}(w') \\ &= L_{z^{l+1}} L_{\delta(st)y_{st}-y_1} \sigma_{st}(w') \end{aligned}$$

Then, assuming (A<sub>n</sub>) and Corollary 4.4, we have

$$\begin{aligned} &\sigma_s^{-1}\widehat{\partial}_n^{(c)}\sigma_s(w) \\ &= \sigma_s^{-1}\widehat{\partial}_n^{(c)} L_{z^{l-1}} L_{\delta(st)y_{st}-y_1} \sigma_{st}(w') \\ &= \sigma_s^{-1} L_{z^{l-1}} (L_{\delta(st)y_{st}-y_1} \widehat{\partial}_n^{(c)} + \widehat{\psi}_n^{(c)}(\delta(st)y_{st} - y_1)) \sigma_{st}(w') \\ &= L_{z_{l,t}} \sigma_{st}^{-1} \widehat{\partial}_n^{(c)} \sigma_{st}(w') + (M_t L_{x^l} + L_{z_{l,t}}) \sigma_{st}^{-1} \widehat{\phi}_{n-1}^{(c)} L_{y_1-\delta(st)y_{st}} \sigma_{st}(w'). \end{aligned}$$

Hence,  $\sigma_s^{-1}\widehat{\partial}_n^{(c)}\sigma_s(w)$  is independent of  $s \in \mu_r$ .

Now, we see that

$$\begin{aligned} &[\sigma_s^{-1} L_{x+\delta(s)y_s}^{-1} \widehat{\partial}_n^{(c)} L_{x+\delta(s)y_s} \sigma_s, L_{z_{l,t}}] \\ &= \sigma_s^{-1} (\widehat{\partial}_n^{(c)} + \widehat{\phi}_{n-1}^{(c)} L_{y_1-\delta(s)y_s}) \sigma_s L_{z_{l,t}} - L_{z_{l,t}} \sigma_s^{-1} (\widehat{\partial}_n^{(c)} + \widehat{\phi}_{n-1}^{(c)} L_{y_1-\delta(s)y_s}) \sigma_s. \end{aligned}$$

Since

$$\begin{aligned} &\sigma_s^{-1} \widehat{\partial}_n^{(c)} \sigma_s L_{z_{l,t}} \\ &= \sigma_s^{-1} \widehat{\partial}_n^{(c)} L_{z^{l-1}} L_{\delta(st)y_{st}-y_1} \sigma_s \\ &= \sigma_s^{-1} L_{z^{l-1}} (L_{\delta(st)y_{st}-y_1} \widehat{\partial}_n^{(c)} + \widehat{\psi}_n^{(c)}(\delta(st)y_{st} - y_1)) \sigma_{st} \\ &= L_{z_{l,t}} \sigma_s^{-1} \widehat{\partial}_n^{(c)} \sigma_{st} + (M_t L_{x^l} + L_{z_{l,t}}) \sigma_{st}^{-1} \widehat{\phi}_{n-1}^{(c)} L_{y_1-\delta(st)y_{st}} \sigma_{st} \\ &= L_{z_{l,t}} \sigma_s^{-1} \widehat{\partial}_n^{(c)} \sigma_s + (M_t L_{x^l} + L_{z_{l,t}}) \sigma_s^{-1} \widehat{\phi}_{n-1}^{(c)} L_{y_1-\delta(s)y_s} \sigma_s, \end{aligned}$$

we have

$$\begin{aligned} &[\sigma_s^{-1} L_{x+\delta(s)y_s}^{-1} \widehat{\partial}_n^{(c)} L_{x+\delta(s)y_s} \sigma_s, L_{z_{l,t}}] \\ &= \sigma_s^{-1} \widehat{\phi}_{n-1}^{(c)} L_{y_1-\delta(s)y_s} \sigma_s L_{z_{l,t}} + M_t L_{x^l} \sigma_s^{-1} \widehat{\phi}_{n-1}^{(c)} L_{y_1-\delta(s)y_s} \sigma_s. \end{aligned}$$

According to Lemma 4.8, this is equivalent to  $[\lambda(\sigma_s^{-1} \widehat{\phi}_{n-1}^{(c)} L_{y_1-\delta(s)y_s} \sigma_s), L_{z_{l,t}}]$ . In addition, we obtain

$$\begin{aligned} &\sigma_s^{-1} L_{x+\delta(s)y_s}^{-1} \widehat{\partial}_n^{(c)} L_{x+\delta(s)y_s} \sigma_s(1) \\ &= \sigma_s^{-1} (\widehat{\partial}_n^{(c)} + \widehat{\phi}_{n-1}^{(c)} L_{y_1-\delta(s)y_s})(1) \\ &= \sigma_s^{-1} \widehat{\phi}_{n-1}^{(c)} L_{y_1-\delta(s)y_s} \sigma_s(1). \end{aligned}$$

Hence, by Lemma 4.12, we have (D<sub>n</sub>).

On the other hand, suppose that (D<sub>n</sub>) is proven. Using (27) and (D<sub>n</sub>), we find

$$(24) \quad \sigma_s^{-1} \widehat{\partial}_n^{(c)} \sigma_s = (\lambda - 1)(\sigma_s^{-1} \widehat{\phi}_{n-1}^{(c)} L_{y_1-\delta(s)y_s} \sigma_s).$$

In addition, based on  $(\beta_n)$ , we have the expression

$$\widehat{\phi}_n^{(c)} = \sum_{i=0}^n \widehat{f}_i^{(c)} L_{z^{n-i}} \quad (\widehat{f}_i^{(c)} \in \mathbb{Q}[\widehat{\partial}_1^{(c)}, \dots, \widehat{\partial}_n^{(c)}]_{(i)}).$$

Hence,

$$\begin{aligned} \sigma_s^{-1} \widehat{\phi}_{n-1}^{(c)} L_{y_1 - \delta(s)y_s} \sigma_s &= \sum_{i=0}^n \sigma_s^{-1} \widehat{f}_i^{(c)} L_{z^{n-i}} L_{y_1 - \delta(s)y_s} \sigma_s \\ &= - \sum_{i=0}^n \sigma_s^{-1} \widehat{f}_i^{(c)} \sigma_s L_{z_{n-i+1,1}}. \end{aligned}$$

By Lemma 4.10 and (28), this is an element of  $\sum_{i=1}^n (\lambda-1)(\mathfrak{W}'_i) L_{z_{n-i+1,1}} + \mathbb{Q} \cdot L_{z_{n,1}}$ . Then, by Lemma 4.9, this is a subset of  $\mathfrak{W}'_{n+1}$ . Thus,  $(C_{n+1})$  is proven.  $\square$

**Theorem 4.14.** *For any  $n \geq 1$  and any  $c \in \mathbb{Q}$ , we have  $\widehat{\partial}_n^{(c)}(\mathcal{A}_{\mu_r}^0) \subset \text{Ker}(\mathcal{L}^{\mathfrak{m}})$ .*

*Proof.* By Proposition 4.13  $(D_n)$ , there exists  $w \in \mathcal{A}_{\{1\},n}^1$  such that

$$\widehat{\partial}_n^{(c)} L_{x+\delta(s)y_s} \sigma_s = L_{x+\delta(s)y_s} \sigma_s \mathcal{H}_w$$

on  $\mathcal{A}_{\mu_r}^1$ . ( $w = L_{x+\delta(s)y_s}^{-1} \widehat{\partial}_n^{(c)}(x + \delta(s)y_s) = L_x^{-1} \varphi \widehat{\partial}_n^{(c)}(x)$ .) Since

$$L_{x+\delta(s)y_s} \sigma_s (\mathcal{A}_{\mu_r, >0}^1) = \mathcal{A}_{\mu_r, >0}^0,$$

we find that

$$\widehat{\partial}_n^{(c)}(\mathcal{A}_{\mu_r, >0}^0) = \widehat{\partial}_n^{(c)} L_{x+\delta(s)y_s} \sigma_s (\mathcal{A}_{\mu_r, >0}^1) = L_{x+\delta(s)y_s} \sigma_s \mathcal{H}_w (\mathcal{A}_{\mu_r, >0}^1).$$

This is a subset of  $L_{x+\delta(s)y_s} \varphi \mathcal{I}M_s(\mathcal{A}_{\mu_r, >0}^1 * \mathcal{A}_{\{1\}, >0}^1)$ . By Corollary 3.21, we obtain

$$\widehat{\partial}_n^{(c)}(\mathcal{A}_{\mu_r, >0}^0) \subset \text{Ker} \mathcal{L}^{\mathfrak{m}}$$

and hence the theorem because  $\widehat{\partial}_n^{(c)}(\mathbb{Q}) = \{0\} \subset \text{Ker} \mathcal{L}^{\mathfrak{m}}$ .  $\square$

Next, we study the properties of the alternative operator  $\partial_n^{(c)}$ , which Kaneko devised by modeling a Hopf algebra developed by A. Connes and H. Moscovici (see [2] for details of the structure), and thereby find that  $\partial_n^{(c)}$  also induces relations of MLV's. The commutativity property of  $\partial_n^{(c)}$  was investigated by Wakiyama [13] in advance.

**Definition 4.15.** *Let  $c$  be a rational number, and let  $H$  be the derivation on  $\mathcal{A}_{\mu_r}$  defined in Definition 4.1. For each integer  $n \geq 1$ , we define a  $\mathbb{Q}$ -linear map  $\partial_n^{(c)}$  from  $\mathcal{A}_{\mu_r}$  to  $\mathcal{A}_{\mu_r}$  by*

$$\partial_n^{(c)} = \frac{1}{(n-1)!} \text{ad}(\theta^{(c)})^{n-1}(\partial_1),$$

where  $\theta^{(c)}$  is the  $\mathbb{Q}$ -linear map defined by  $\theta^{(c)}(u) = \theta(u)$  ( $u = x$  or  $y_s$ ) and the rule

$$\theta^{(c)}(ww') = \theta^{(c)}(w)w' + w\theta^{(c)}(w') + c\partial_1(w)H(w')$$

for any  $w, w' \in \mathcal{A}_{\mu_r}$ .

The only difference between  $\widehat{\partial}_n^{(c)}$  and  $\theta^{(c)}$  is the order of  $H$  and  $\partial_1$  appearing in the right-hand side of their recursive rules.

**Lemma 4.16.** *For any rational number  $c$ , we have*

$$\theta^{(c)} = \widehat{\theta}^{(-c)} + c\partial_1(H-1).$$

*Proof.* We immediately see that the images of generators  $x$  and  $y_s$  ( $s \in \mu_r$ ) of both sides coincide. For  $w, w' \in \mathcal{A}_{\mu_r}$ ,

$$\begin{aligned} & (\widehat{\theta}^{(-c)} + c\partial_1(H-1))(ww') \\ = & \widehat{\theta}^{(-c)}(w)w' + w\widehat{\theta}^{(-c)}(w') - cH(w)\partial_1(w') + c(\partial_1H(w)w' \\ & + H(w)\partial_1(w') + \partial_1(w)H(w') + w\partial_1H(w')) - c(\partial_1(w)w' + w\partial_1(w')) \\ = & (\widehat{\theta}^{(-c)} + c\partial_1(H-1))(w)w' + w(\widehat{\theta}^{(-c)} + c\partial_1(H-1))(w') + c\partial_1(w)H(w'). \end{aligned}$$

Hence, the recursive rules also coincide, and the Lemma is proven.  $\square$

**Proposition 4.17.** *For any positive integer  $n$  and any rational number  $c$ , we have*

$$\partial_n^{(c)} \in \mathbb{Q}[\widehat{\partial}_1^{(-c)}, \dots, \widehat{\partial}_n^{(-c)}]_{(n)}.$$

*Proof.* The proposition holds for  $n = 1$  because  $\partial_1^{(c)} = \widehat{\partial}_1^{(-c)} = \partial_1$ . Suppose that the proposition is proven for  $n$ . Using Lemma 4.16 and Proposition 4.5, we obtain

$$\begin{aligned} n\partial_{n+1}^{(c)} &= [\theta^{(c)}, \partial_n^{(c)}] \\ &= [\widehat{\theta}^{(-c)} + c\partial_1(H-1), \partial_n^{(c)}] \\ &= [\widehat{\theta}^{(-c)}, \partial_n^{(c)}] + c(n-1)\partial_1\partial_n^{(c)}. \end{aligned}$$

Hence, by induction the Proposition holds for  $n + 1$ .  $\square$

**Corollary 4.18.** *For any rational numbers  $c, c'$ , and any positive integers  $n, m$ , we have*

$$[\partial_n^{(c)}, \widehat{\partial}_m^{(c')}] = 0.$$

*Proof.* Immediately from Proposition 4.5 and 4.17.  $\square$

By Proposition 4.17 and Theorem 4.14, we have

$$(25) \quad \partial_n^{(c)}(\mathcal{A}_{\mu_r}^0) \subset \text{Ker} \mathcal{L}^{\mathfrak{m}},$$

which gives the same class as Theorem 4.14.

**4.3. On the Derivation Relation.** Let  $\widehat{\mathcal{A}}_{\mu_r}$  denote the completion of  $\mathcal{A}_{\mu_r}$ . We put  $\widehat{\Delta}$  as

$$(26) \quad \widehat{\Delta} = \exp\left(\sum_{n=1}^{\infty} \frac{\partial_n}{n}\right)$$

which has been introduced in [1]. Then,  $\widehat{\Delta}$  is an automorphism on  $\widehat{\mathcal{A}}_{\mu_r}$  that satisfies

$$\widehat{\Delta}(x) = x \frac{1}{1-y_1}, \quad \widehat{\Delta}(y_s) = (x+y_s) \frac{1}{1-y_1+y_s}$$

for any  $s \in \mu_r$ . We find that

$$\widehat{\Delta}(x + \delta(s)y_s) = (x + \delta(s)y_s) \frac{1}{1-y_1+\delta(s)y_s}, \quad \widehat{\Delta}(z) = z,$$

or, in other words,

$$\widehat{\Delta}L_{x+\delta(s)y_s} = L_{x+\delta(s)y_s}L_{\frac{1}{1-y_1+\delta(s)y_s}}\widehat{\Delta}, \quad L_z\widehat{\Delta} = \widehat{\Delta}L_z.$$

Now, let  $\Phi$  be

$$\Phi(w) = (1 + y_1) \left( \frac{1}{1 + y_1} * w \right) \quad (w \in \hat{\mathcal{A}}_{\mu_r}^1)$$

as in [1], where  $\hat{\mathcal{A}}_{\mu_r}^1$  is the completion of  $\mathcal{A}_{\mu_r}^1$ . We see that  $\Phi$  can be extended to an automorphism on  $\hat{\mathcal{A}}_{\mu_r}$  that satisfies

$$\Phi(x) = x, \quad \Phi(x + y_s) = y_s \frac{1}{1 + y_1} + x - xy_s \frac{1}{1 + y_1}$$

and hence

$$\Phi(y_s) = (1 - x)y_s \frac{1}{1 + y_1}.$$

**Lemma 4.19.** *Let  $s \in \mu_r$ . We have  $\varphi \mathcal{I}M_s \Phi = \hat{\Delta} \varphi \mathcal{I}M_s$  on  $\mathcal{A}_{\mu_r}$ .*

*Proof.* We need only show the equality of the images of general monomials in  $\mathcal{A}_{\mu_r}$  on both sides.

$$\begin{aligned} & \hat{\Delta} \varphi \mathcal{I}M_s(z_{k_1, s_1} \cdots z_{k_n, s_n} x^l) \\ &= \hat{\Delta} \varphi(z_{k_1, s s_1} \cdots z_{k_n, s s_1 \cdots s_n} x^l) \\ &= \hat{\Delta} \left( z^{k_1-1} (\delta(ss_1)y_{ss_1} - y_1) \cdots z^{k_n-1} (\delta(ss_1 \cdots s_n)y_{ss_1 \cdots s_n} - y_1) z^l \right) \\ &= z^{k_1-1} \left( (x + \delta(ss_1)y_{ss_1}) \frac{1}{1 - y_1 + \delta(ss_1)y_{ss_1}} - z \right) \\ & \quad \cdots z^{k_n-1} \left( (x + \delta(ss_1 \cdots s_n)y_{ss_1 \cdots s_n}) \frac{1}{1 - y_1 + \delta(ss_1 \cdots s_n)y_{ss_1 \cdots s_n}} - z \right) z^l. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \varphi \mathcal{I}M_s \Phi(z_{k_1, s_1} \cdots z_{k_n, s_n} x^l) \\ &= \varphi \mathcal{I}M_s \left( x^{k_1-1} (1 - x)y_{s_1} \frac{1}{1 + y_1} \cdots x^{k_n-1} (1 - x)y_{s_n} \frac{1}{1 + y_1} x^l \right) \\ &= \varphi \left( x^{k_1-1} (1 - x)y_{s s_1} \frac{1}{1 + y_{s s_1}} \cdots x^{k_n-1} (1 - x)y_{s s_1 \cdots s_n} \frac{1}{1 + y_{s s_1 \cdots s_n}} x^l \right) \\ &= z^{k_1-1} (1 - z) (\delta(ss_1)y_{ss_1} - y_1) \frac{1}{1 - y_1 + \delta(ss_1)y_{ss_1}} \\ & \quad \cdots z^{k_n-1} (1 - z) (\delta(ss_1 \cdots s_n)y_{ss_1 \cdots s_n} - y_1) \frac{1}{1 - y_1 + \delta(ss_1 \cdots s_n)y_{ss_1 \cdots s_n}} z^l. \end{aligned}$$

We can easily see that

$$(x + \delta(s)y_s) \frac{1}{1 - y_1 + \delta(s)y_s} = z + (1 - z) (\delta(s)y_s - y_1) \frac{1}{1 - y_1 + \delta(s)y_s}.$$

Hence, the lemma is proven.  $\square$

**Proposition 4.20.** *Let  $s \in \mu_r$ . Then, we have*

$$\mathcal{H}_{\frac{1}{1+y_1}} = M_{\frac{1}{s}} \mathcal{I}^{-1} \varphi L_{x+\delta(s)y_s} \hat{\Delta} L_{x+\delta(s)y_s} \varphi \mathcal{I}M_s$$

on  $\mathcal{A}_{\mu_r}^1$ .

*Proof.* Using Lemma 4.19, we have

$$\begin{aligned}
\text{RHS} &= M_{\frac{1}{s}} \mathcal{I}^{-1} \varphi L_{\frac{1}{1-y_1+\delta(f)y_f}} \widehat{\Delta}_{1-X} \varphi \mathcal{I} M_s \\
&= M_{\frac{1}{s}} \mathcal{I}^{-1} L_{\frac{1}{1+y_s}} \varphi \widehat{\Delta}_{1-X} \varphi \mathcal{I} M_s \\
&= L_{\frac{1}{1+y_1}} M_{\frac{1}{s}} \mathcal{I}^{-1} \varphi \widehat{\Delta}_{1-X} \varphi \mathcal{I} M_s \\
&= L_{\frac{1}{1+y_1}} \Phi_{-y_1} = \text{LHS}.
\end{aligned}$$

□

Based on equation (26), Proposition 4.20, and Corollary 3.21, we again obtain the derivation relation

$$\partial_n(\mathcal{A}_{\mu_r}^0) \subset \text{Ker} \mathcal{L}^{\mathfrak{m}}.$$

## 5. PROOFS OF LEMMATA

*Proof of Lemma 3.12.* If (i) and (ii) are proven, then (i)' and (ii)' also hold by substituting  $d_*^{-1}(w), d_*^{-1}(w')$  in place of  $w, w'$  and by applying  $d_*^{-1}$  to both sides. Therefore, we show (i) and (ii).

(i) By induction on the depth of a word. We denote the depth of a word  $w$  by  $\text{dep}(w)$ . When  $\text{dep}(w) = 1$ , set  $w = z_{k,s}, w' = z_{l,t}v$ . If  $\text{dep}(v) = 0$ , then

$$\begin{aligned}
\text{LHS} &= \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(z_{k,s} \bar{*} z_{l,t}) \\
&= \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(z_{k,s} z_{l,t} + z_{l,t} z_{k,s} - z_{k+l,st}) \\
&= \mathcal{I}^{-1} d_{\mathfrak{m}} (z_{k,s} z_{l,st} + z_{l,t} z_{k,st} - z_{k+l,st}) \\
&= \mathcal{I}^{-1} ((z_{k,s} + x^k) z_{l,st} + (z_{l,t} + x^l) z_{k,st} - z_{k+l,st}) \\
&= \mathcal{I}^{-1} (z_{k,s} z_{l,st} + z_{l,t} z_{k,st} + z_{k+l,st}) = \text{RHS}.
\end{aligned}$$

Hence, the assertion is proven.

If  $\text{dep}(v) > 0$ , then

$$\begin{aligned}
\text{LHS} &= \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(z_{k,s} \bar{*} z_{l,t}v) \\
&= \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(z_{k,s} z_{l,t}v + z_{l,t}(z_{k,s} \bar{*} v) - z_{k+l,st}v) \\
&= \mathcal{I}^{-1} d_{\mathfrak{m}} L_{z_{k,s}} N_{\frac{1}{s}} L_{z_{l,t}} N_{\frac{1}{t}} \mathcal{I}(v) + \mathcal{I}^{-1} d_{\mathfrak{m}} L_{z_{l,t}} N_{\frac{1}{t}} \mathcal{I}(z_{k,s} \bar{*} v) \\
&\quad - \mathcal{I}^{-1} d_{\mathfrak{m}} L_{z_{k+l,st}} N_{\frac{1}{st}} \mathcal{I}(v) \\
&= \mathcal{I}^{-1} L_{z_{k,s}+x^k} N_{\frac{1}{s}} L_{z_{l,t}+x^l} N_{\frac{1}{t}} d_{\mathfrak{m}} \mathcal{I}(v) + \mathcal{I}^{-1} L_{z_{l,t}+x^l} N_{\frac{1}{t}} d_{\mathfrak{m}} \mathcal{I}(z_{k,s} \bar{*} v) \\
&\quad - \mathcal{I}^{-1} L_{z_{k+l,st}+x^{k+l}} N_{\frac{1}{st}} d_{\mathfrak{m}} \mathcal{I}(v) \\
&= L_{z_{k,s}} L_{z_{l,t}} \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(v) + L_{z_{k,s}} L_{x^l} M_t \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(v) \\
&\quad + L_{x^k} L_{z_{l,st}} \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(v) + L_{x^{k+l}} M_{st} \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(v) \\
&\quad + L_{z_{l,t}} (z_{k,s} * \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(v)) + L_{x^l} M_t (z_{k,s} \bar{*} \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(v)) \\
&\quad - L_{z_{k+l,st}} \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(v) - L_{x^{k+l}} M_{st} \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(v), \\
\text{RHS} &= z_{k,s} * \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(z_{l,t}v) \\
&= z_{k,s} * (L_{z_{l,t}} \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(v) + L_{x^l} M_t \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(v)) \\
&= L_{z_{k,s}} L_{z_{l,t}} \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(v) + L_{z_{l,t}} (z_{k,s} * \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(v)) \\
&\quad + L_{z_{k+l,st}} \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(v) + z_{k,s} * (L_{x^l} M_t \mathcal{I}^{-1} d_{\mathfrak{m}} \mathcal{I}(v)).
\end{aligned}$$

Hence, we must show that

$$L_{z_{k,s}} L_{x^l} M_t(V) + L_{x^l} M_t(z_{k,s} * V) = L_{z_{k+l,st}}(V) + z_{k,s} * (L_{x^l} M_t(V)).$$

Setting  $V = z_{k_1,s_1} \cdots z_{k_1,s_n}$ , we have

$$\begin{aligned} \text{LHS} &= z_{k,s} z_{k_1+l,s_1} t z_{k_2,s_2} \cdots z_{k_1,s_n} + L_{x^l} M_t(z_{k,s} z_{k_1,s_1} \cdots z_{k_1,s_n} \\ &\quad + z_{k_1,s_1}(z_{k,s} * z_{k_1,s_2} \cdots z_{k_1,s_n}) + z_{k+k_1,ss_1} z_{k_1,s_2} \cdots z_{k_1,s_n}), \\ \text{RHS} &= z_{k+l,st} z_{k_1,s_1} \cdots z_{k_1,s_n} + z_{k,s} * z_{k_1+l,s_1} t z_{k_1,s_2} \cdots z_{k_1,s_n} \\ &= z_{k+l,st} z_{k_1,s_1} \cdots z_{k_1,s_n} + z_{k,s} z_{k_1+l,s_1} t z_{k_1,s_2} \cdots z_{k_1,s_n} \\ &\quad + z_{k_1+l,s_1} t (z_{k,s} * z_{k_1,s_2} \cdots z_{k_1,s_n}) + z_{k+k_1+l,ss_1} t z_{k_1,s_2} \cdots z_{k_1,s_n}. \end{aligned}$$

Thus, we obtain LHS=RHS.

If  $\text{dep}(w) > 1$ , we can set  $w = z_{k,s}u, w' = z_{l,t}v$  ( $\text{dep}(u), \text{dep}(v) > 0$ ). Then

$$\begin{aligned} \text{LHS} &= \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(z_{k,s}u \bar{*} z_{l,t}v) \\ &= \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(z_{k,s}(u \bar{*} z_{l,t}v) + z_{l,t}(z_{k,s}u \bar{*} v) - z_{k+l,st}(u \bar{*} v)) \\ &= \mathcal{I}^{-1} d_{\mathbf{m}} L_{z_{k,s}} M_{\frac{1}{s}} \mathcal{I}(u \bar{*} z_{l,t}v) + \mathcal{I}^{-1} d_{\mathbf{m}} L_{z_{l,t}} M_{\frac{1}{t}} \mathcal{I}(z_{k,s}u \bar{*} v) \\ &\quad - \mathcal{I}^{-1} d_{\mathbf{m}} L_{z_{k+l,st}} M_{\frac{1}{st}} \mathcal{I}(u \bar{*} v) \\ &= \mathcal{I}^{-1} L_{z_{k,s}+x^k} M_{\frac{1}{s}} d_{\mathbf{m}} \mathcal{I}(u \bar{*} z_{l,t}v) + \mathcal{I}^{-1} L_{z_{l,t}+x^l} M_{\frac{1}{t}} d_{\mathbf{m}} \mathcal{I}(z_{k,s}u \bar{*} v) \\ &\quad - \mathcal{I}^{-1} L_{z_{k+l,st}+x^{k+l}} M_{\frac{1}{st}} d_{\mathbf{m}} \mathcal{I}(u \bar{*} v) \\ &= L_{z_{k,s}} \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u \bar{*} z_{l,t}v) + L_{x^l} M_s \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u \bar{*} z_{l,t}v) \\ &\quad + L_{z_{l,t}} \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(z_{k,s}u \bar{*} v) + L_{x^k} M_t \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(z_{k,s}u \bar{*} v) \\ &\quad - L_{z_{k+l,st}} \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u \bar{*} v) - L_{x^{k+l}} M_{st} \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u \bar{*} v) \\ &= L_{z_{k,s}} (\mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) * (L_{z_{l,t}} \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v) + L_{x^l} M_t \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v))) \\ &\quad + L_{x^k} M_s (\mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) * (L_{z_{l,t}} \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v) + L_{x^l} M_t \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v))) \\ &\quad + L_{z_{l,t}} ((L_{z_{k,s}} \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) + L_{x^k} M_s \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u)) * \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v)) \\ &\quad + L_{x^l} M_t ((L_{z_{k,s}} \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) + L_{x^k} M_s \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u)) * \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v)) \\ &\quad - L_{z_{k+l,st}} (\mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) * \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v)) \\ &\quad - L_{x^{k+l}} M_{st} (\mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) * \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v)), \\ \text{RHS} &= ((L_{z_{k,s}} + L_{x^k} M_s) \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u)) * ((L_{z_{l,t}} + L_{x^l} M_t) \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v)) \\ &= L_{z_{k,s}} (\mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) * L_{z_{l,t}} \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v)) \\ &\quad + L_{z_{l,t}} (L_{z_{k,s}} \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) * \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v)) \\ &\quad + L_{z_{k+l,st}} (\mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) * \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v)) \\ &\quad + L_{x^k} M_s \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) * L_{z_{l,t}} \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v) \\ &\quad + L_{z_{k,f}} \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) * L_{x^l} M_t \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v) \\ &\quad + L_{x^k} M_s \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) * L_{x^l} M_t \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v). \end{aligned}$$

Hence, we must show that

$$\begin{aligned}
& L_{x^k} M_s(V * (L_{z_{l,t}} + L_{x^l} M_t)(W)) + L_{z_{k,s}}(V * L_{x^l} M_t(W)) \\
& + L_{x^l} M_t((L_{z_{k,s}} + L_{x^k} M_s)(V) * W) + L_{z_{l,t}}(L_{x^k} M_s(V) * W) \\
& - (L_{z_{k+l,st}} + L_{x^{k+l}} M_{st})(V * W) \\
& = L_{z_{k,s}}(V) * L_{x^l} M_t(W) + L_{x^k} M_s(V) * L_{z_{l,t}}(W) \\
& + L_{x^k} M_s(V) * L_{x^l} M_t(W) + L_{z_{k+l,st}}(V * W).
\end{aligned}$$

Setting  $V = z_{p,i}v, W = z_{q,j}w$ ,

$$\begin{aligned}
\text{LHS} = & L_{x^k} M_s(L_{z_{p,i}}(v * L_{z_{l,t}}(W)) + L_{z_{l,t}}(V * W) + L_{z_{p+l,ti}}(v * W) \\
& + L_{z_{p,i}}(v * L_{x^l} M_t(W)) + L_{z_{l+q,tj}}(V * w) + L_{z_{l+p+q,tij}}(v * w)) \\
& + L_{x^l} M_t(L_{z_{k,s}}(V * W) + L_{z_{q,j}}(L_{z_{k,s}}(V) * w) + L_{z_{k+q,sj}}(V * w)) \\
& + L_{z_{k+p,si}}(v * W) + L_{z_{q,j}}(L_{x^l} M_s(V) * w) + L_{z_{k+p+q,sij}}(v * w)) \\
& + L_{z_{k,s}}(V * L_{x^l} M_t(W)) + L_{z_{l,t}}(L_{x^k} M_s(V) * W) \\
& - L_{z_{k+l,st}}(V * W) - L_{x^{k+l}} M_{st}(L_{z_{p,i}}(v * W) \\
& + L_{z_{q,j}}(V * w) + L_{z_{p+q,ij}}(v * w))
\end{aligned}$$

and

$$\begin{aligned}
\text{RHS} = & L_{z_{k,s}}(V * L_{x^l} M_t(W)) + L_{z_{l+q,tj}}(L_{z_{k,s}}(V) * W) \\
& + L_{z_{k+l+q,stj}}(V * w) + L_{z_{k+p,si}}(v * L_{z_{l,t}}(W)) \\
& + L_{z_{l,t}}(L_{x^k} M_s(V) * W) + L_{z_{k+l+p,sti}}(v * W) \\
& + L_{z_{k+p,si}}(v * L_{x^l} M_t(W)) + L_{z_{l+q,tj}}(L_{x^k} M_s(V) * w) \\
& + L_{z_{k+l+p+q,stij}}(v * w) + L_{z_{k+l,st}}(V * W).
\end{aligned}$$

Thus, we obtain LHS=RHS.

(ii) Again, by the induction on the depth of a word. If  $\text{dep}(w) = 1$ , set  $w = z_{k,s}, w' = z_{l,t}v$ . If  $\text{dep}(v) = 0$ , then both sides become  $z_{k+l,st}$ , and the assertion holds.

If  $\text{dep}(v) > 0$ , then

$$\text{LHS} = \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(z_{k+l,st}v) = (L_{z_{k+l,st}} + L_{x^{k+l}} M_{st}) \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v),$$

$$\begin{aligned}
\text{RHS} &= z_{k,s} \dot{*} (L_{z_{l,t}} + L_{x^l} M_t) \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v) \\
&= L_{z_{k+l,st}} \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v) + z_{k,s} \dot{*} L_{x^l} M_t \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v).
\end{aligned}$$

Hence, we must show that

$$z_{k,s} \dot{*} L_{x^l} M_t(V) = L_{x^{k+l}} M_{st}(V)$$

Setting  $V = z_{p,i}v$ , we have

$$\text{LHS} = z_{k+l+p,sti}v = \text{RHS}.$$

Hence, the assertion is proven.

If  $\text{dep}(w) > 1$ , we can set  $w = z_{k,s}u, w' = z_{l,t}v$  ( $\text{dep}(u), \text{dep}(v) > 0$ ). Then

$$\begin{aligned}
\text{LHS} &= \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(z_{k+l,st}(u \bar{*} v)) \\
&= (L_{z_{k+l,st}} + L_{x^{k+l}} M_{st}) \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u \bar{*} v) \\
&= (L_{z_{k+l,st}} + L_{x^{k+l}} M_{st})(\mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) * \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v)),
\end{aligned}$$



$$\begin{aligned}
 \text{RHS} &= (L_{z_{k,s}} + L_{x^k} M_s) \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) \dot{*} (L_{z_{l,t}} + L_{x^l} M_t) \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v) \\
 &= L_{z_{k+l,st}} (\mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) * \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v)) \\
 &\quad + L_{z_{k,s}} \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) \dot{*} L_{x^l} M_t \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v) \\
 &\quad + L_{x^k} M_s \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) \dot{*} L_{z_{l,t}} \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v) \\
 &\quad + L_{x^k} M_s \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(u) \dot{*} L_{x^l} M_t \mathcal{I}^{-1} d_{\mathbf{m}} \mathcal{I}(v).
 \end{aligned}$$

Hence, we must show that

$$\begin{aligned}
 &L_{z_{k,s}}(V) \dot{*} L_{x^l} M_t(W) + L_{x^k} M_s(V) \dot{*} L_{z_{l,t}}(W) \\
 &\quad + L_{x^k} M_s(V) \dot{*} L_{x^l} M_t(W) = L_{x^{k+l}} M_{st}(V * W)
 \end{aligned}$$

Setting  $V = z_{p,i}v$ ,  $W = z_{q,j}w$ , we have

$$\begin{aligned}
 \text{LHS} &= z_{k+l+q,stj}(V * w) + z_{k+l+p,sti}(v * W) + z_{k+l+p+q,stij}(v * w) \\
 &= L_{x^{k+l}} M_{st}(z_{q,j}(V * w) + z_{p,i}(v * W) + z_{p+q,ij}(v * w)) \\
 &= \text{RHS}.
 \end{aligned}$$

This completes the proof.  $\square$

*Proof of Lemma 3.13.* First, we show the identity

$$(27) \quad F_s L_{z_{k,t}} = L_{z_{k, \frac{1-\delta(st)st}{1-\delta(s)s}}} F_{st}$$

for  $s, t \in \Lambda$ . If  $s = 1$ , then

$$\begin{aligned}
 F_1 L_{z_{k,t}} &= \mathcal{I}^{-1} \iota \mathcal{I} L_{z_{k,t}} = \mathcal{I}^{-1} \iota L_{z_{k,t}} \mathcal{I} M_t = \mathcal{I}^{-1} (\delta(t) L_{z_{k,1-t}} + (1 - \delta(t)) L_{z_{k,1}}) \iota \mathcal{I} M_t \\
 &= \delta(t) L_{z_{k,1-t}} M_{\frac{1}{1-t}} \mathcal{I}^{-1} \iota \mathcal{I} M_t + (1 - \delta(t)) L_{z_{k,1}} \mathcal{I}^{-1} \iota \mathcal{I} M_t = L_{z_{k,1-\delta(t)t}} F_t.
 \end{aligned}$$

If  $s \neq 1$ , then

$$\begin{aligned}
 F_s L_{z_{k,t}} &= M_{\frac{1}{1-s}} \mathcal{I}^{-1} \iota \mathcal{I} M_s L_{z_{k,t}} = M_{\frac{1}{1-s}} \mathcal{I}^{-1} \iota L_{z_{k,st}} \mathcal{I} M_{st} \\
 &= M_{\frac{1}{1-s}} \mathcal{I}^{-1} (\delta(st) L_{z_{k,1-st}} + (1 - \delta(st)) L_{z_{k,1}}) \iota \mathcal{I} M_{st} \\
 &= \delta(st) L_{z_{k, \frac{1-st}{1-s}}} M_{\frac{1}{1-st}} \mathcal{I}^{-1} \iota \mathcal{I} M_{st} + (1 - \delta(st)) L_{z_{k, \frac{1}{1-s}}} \mathcal{I}^{-1} \iota \mathcal{I} M_{st} = L_{z_{k, \frac{1-\delta(st)st}{1-s}}} F_{st}.
 \end{aligned}$$

Therefore, we have the assertion (27).

To prove Lemma, we must show that

$$(28) \quad F_s(w) \bar{*} w' = F_s(w \bar{*} w')$$

based on Lemma 3.12 i)' . The equality is proven by induction on the total depth of  $w$  and  $w'$ . It is simple to show that  $\text{dep}(w) = 1$  or  $\text{dep}(w') = 1$ , hence (28) holds for  $\text{dep}(w) + \text{dep}(w') \leq 1$ . Set  $w = z_{k_1,t} w_1$ ,  $w' = z_{k'_1,1} w'_1$  ( $w \in \mathcal{A}_{\Lambda, >0}^1$ ,  $w' \in \mathcal{A}_{\{1\}, >0}^1$ ).

According to the identity (27) and the induction hypothesis, we have

$$\begin{aligned}
 F_s(w) \bar{*} w' &= L_{z_{k_1, \frac{1-\delta(st)st}{1-\delta(s)s}}} F_{st}(w_1) \bar{*} L_{z_{k'_1,1}}(w'_1) \\
 &= L_{z_{k_1, \frac{1-\delta(st)st}{1-\delta(s)s}}} (F_{st}(w_1) \bar{*} L_{z_{k'_1,1}} w'_1) + L_{z_{k'_1,1}} (F_s(w) \bar{*} w'_1) - L_{z_{k_1+k'_1, \frac{1-\delta(st)st}{1-\delta(s)s}}} (F_{st}(w_1) \bar{*} w'_1) \\
 &= L_{z_{k_1, \frac{1-\delta(st)st}{1-\delta(s)s}}} (F_{st}(w_1 \bar{*} L_{z_{k'_1,1}} w'_1)) + L_{z_{k'_1,1}} (F_s(w \bar{*} w'_1)) - L_{z_{k_1+k'_1, \frac{1-\delta(st)st}{1-\delta(s)s}}} (F_{st}(w_1 \bar{*} w'_1)) \\
 &= F_s L_{z_{k_1,t}} (w_1 \bar{*} z_{k'_1,1} w'_1) + F_s L_{z_{k'_1,1}} (w \bar{*} w'_1) - F_s L_{z_{k_1+k'_1,t}} (w_1 \bar{*} w'_1) \\
 &= F_s(w \bar{*} w').
 \end{aligned}$$

Hence, (28) holds. (Actually, (28) holds if  $\bar{*}$  is changed to  $*$ .)  $\square$

*Proof of Lemma 4.3.* The proof is given by induction on  $n$ . Since  $\widehat{\psi}_1^{(c)}(x) = L_x L_{y_1}$ ,  $\widehat{\psi}_1^{(c)}(y_1) = -L_x L_{y_1}$  and  $\widehat{\psi}_1^{(c)}(y_s) = -L_x L_{y_s} + L_{y_s} L_{y_1-y_s}$  ( $s \neq 1$ ), we obtain the lemma for  $n = 1$  by setting  $\widehat{\phi}_0^{(c)} = \text{id}_{A_{\mu_r}}$ .

Suppose that the lemma is proven for  $n$ . By the recursive rule of  $\widehat{\psi}_n^{(c)}(u)$  and the induction hypothesis, we have

$$\begin{aligned} n\widehat{\psi}_{n+1}^{(c)}(u) &= [\widehat{\theta}^{(c)}, \widehat{\psi}_n^{(c)}(u)] - \frac{1}{2}(L_z \widehat{\psi}_n^{(c)}(u) + \widehat{\psi}_n^{(c)}(u)L_z) - c\widehat{\psi}_n^{(c)}(u)\partial_1 \\ &= [\widehat{\theta}^{(c)}, (-1)^{\nu(u)}(L_x \widehat{\phi}_{n-1}^{(c)} L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_u \widehat{\phi}_{n-1}^{(c)} L_{u-y_1})] \\ &\quad - \frac{1}{2}(-1)^{\nu(u)}L_z(L_x \widehat{\phi}_{n-1}^{(c)} L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_u \widehat{\phi}_{n-1}^{(c)} L_{u-y_1}) \\ &\quad - \frac{1}{2}(-1)^{\nu(u)}(L_x \widehat{\phi}_{n-1}^{(c)} L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_u \widehat{\phi}_{n-1}^{(c)} L_{u-y_1})L_z \\ &\quad - c(-1)^{\nu(u)}(L_x \widehat{\phi}_{n-1}^{(c)} L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_u \widehat{\phi}_{n-1}^{(c)} L_{u-y_1})\partial_1. \end{aligned}$$

Write the leading Lie bracket term as the sum

$$\begin{aligned} &(-1)^{\nu(u)}\left\{[\widehat{\theta}^{(c)}, L_x]\widehat{\phi}_{n-1}^{(c)} L_{y_1+\nu(u)(u-y_1)} + L_x[\widehat{\theta}^{(c)}, \widehat{\phi}_{n-1}^{(c)}]L_{y_1+\nu(u)(u-y_1)}\right. \\ &\quad + L_x \widehat{\phi}_{n-1}^{(c)}[\widehat{\theta}^{(c)}, L_{y_1+\nu(u)(u-y_1)}] + \nu(u)([\widehat{\theta}^{(c)}, L_u]\widehat{\phi}_{n-1}^{(c)} L_{u-y_1} \\ &\quad \left.+ L_u[\widehat{\theta}^{(c)}, \widehat{\phi}_{n-1}^{(c)}]L_{u-y_1} + L_u \widehat{\phi}_{n-1}^{(c)}[\widehat{\theta}^{(c)}, L_{u-y_1}]\right\}. \end{aligned}$$

Using the identity  $[\widehat{\theta}^{(c)}, L_u] = L_{\widehat{\theta}^{(c)}(u)} + cL_u\partial_1$ , which can be easily shown, we have

$$\begin{aligned} n\widehat{\psi}_{n+1}^{(c)}(u) &= (-1)^{\nu(u)}\left\{(L_{\widehat{\theta}^{(c)}(u)} + cL_x\partial_1)\widehat{\phi}_{n-1}^{(c)} L_{y_1+\nu(u)(u-y_1)} + L_x[\widehat{\theta}^{(c)}, \widehat{\phi}_{n-1}^{(c)}]L_{y_1+\nu(u)(u-y_1)}\right. \\ &\quad + L_x \widehat{\phi}_{n-1}^{(c)}(L_{\widehat{\theta}^{(c)}(y_1+\nu(u)(u-y_1))} + cL_{y_1+\nu(u)(u-y_1)}\partial_1) \\ &\quad + \nu(u)((L_{\widehat{\theta}^{(c)}(u)} + cL_u\partial_1)\widehat{\phi}_{n-1}^{(c)} L_{u-y_1} \\ &\quad + L_u[\widehat{\theta}^{(c)}, \widehat{\phi}_{n-1}^{(c)}]L_{u-y_1} + L_u \widehat{\phi}_{n-1}^{(c)}(L_{\widehat{\theta}^{(c)}(u-y_1)} + cL_{u-y_1}\partial_1)) \\ &\quad - \frac{1}{2}L_z(L_x \widehat{\phi}_{n-1}^{(c)} L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_u \widehat{\phi}_{n-1}^{(c)} L_{u-y_1}) \\ &\quad - \frac{1}{2}(L_x \widehat{\phi}_{n-1}^{(c)} L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_u \widehat{\phi}_{n-1}^{(c)} L_{u-y_1})L_z \\ &\quad \left. - c(L_x \widehat{\phi}_{n-1}^{(c)} L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_u \widehat{\phi}_{n-1}^{(c)} L_{u-y_1})\partial_1\right\}. \end{aligned}$$

Considering  $\theta^{(c)}(u) = \frac{1}{2}(uz + zu)$ ,

$$\begin{aligned} n\widehat{\psi}_{n+1}^{(c)}(u) &= (-1)^{\nu(u)}\left\{\frac{1}{2}L_x L_z \widehat{\phi}_{n-1}^{(c)} L_{y_1+\nu(u)(u-y_1)} + cL_x \partial_1 \widehat{\phi}_{n-1}^{(c)} L_{y_1+\nu(u)(u-y_1)}\right. \\ &\quad + L_x[\widehat{\theta}^{(c)}, \widehat{\phi}_{n-1}^{(c)}]L_{y_1+\nu(u)(u-y_1)} + \frac{1}{2}L_x \widehat{\phi}_{n-1}^{(c)} L_z L_{y_1+\nu(u)(u-y_1)} \\ &\quad + \nu(u)\left(\frac{1}{2}L_u L_z \widehat{\phi}_{n-1}^{(c)} L_{u-y_1} + cL_u \partial_1 \widehat{\phi}_{n-1}^{(c)} L_{u-y_1}\right. \\ &\quad \left.+ L_u[\widehat{\theta}^{(c)}, \widehat{\phi}_{n-1}^{(c)}]L_{u-y_1} + \frac{1}{2}L_u \widehat{\phi}_{n-1}^{(c)} L_z L_{u-y_1}\right)\left.\right\}. \end{aligned}$$

By setting

$$(29) \quad \widehat{\phi}_n^{(c)} = \frac{1}{n}\left([\widehat{\theta}^{(c)}, \widehat{\phi}_{n-1}^{(c)}] + \frac{1}{2}(L_z \widehat{\phi}_{n-1}^{(c)} + \widehat{\phi}_{n-1}^{(c)} L_z) + c\partial_1 \widehat{\phi}_{n-1}^{(c)}\right),$$

we have  $n\widehat{\psi}_{n+1}^{(c)}(u) = (-1)^{\nu(u)}n(L_x \widehat{\phi}_n^{(c)} L_{y_1+\nu(u)(u-y_1)} + \nu(u)L_u \widehat{\phi}_n^{(c)} L_{u-y_1})$ . This shows the lemma.  $\square$

## APPENDIX: TABLES

We give the maximal number of linearly independent relations supplied by each set of relations among MLV's for each  $\mu_r$  with  $1 \leq r \leq 6$ . The lineup is the derivation relation (" $\partial_n$ "), the extended derivation relation (" $\partial_n^{(c)}$ ") for all  $c \in \mathbb{Q}$ , the linear part of Corollary 3.21 ("lin.") sequentially from the top. (We have already proven " $\partial_n \subset \partial_n^{(c)} \subset \text{lin.}$ ") Computations were performed using Risa/Asir, an open-source general computer algebra system.

 $r = 1$  (MZV case)

weight	3	4	5	6	7	8	9	10	11	12	13	14
$\partial_n$	1	2	5	10	22	44	90	181	363	727	1456	2912
$\partial_n^{(c)}$	1	2	5	10	23	46	98	200	410	830	1679	...
lin.	1	2	5	10	23	46	98	200	413	838	1713	...
$\#\{\text{index set}\}$	2	4	8	16	32	64	128	256	512	1024	2048	4096

 $r = 2$ 

weight	3	4	5	6	7	8
$\partial_n$	4	14	46	140	426	1280
$\partial_n^{(c)}$	4	14	48	150	464	1402
lin.	4	14	48	150	468	1422
$\#\{\text{index set}\}$	12	36	108	324	972	2916

 $r = 3$ 

weight	3	4	5	6
$\partial_n$	9	42	177	714
$\partial_n^{(c)}$	9	42	183	750
lin.	9	42	183	750
$\#\{\text{index set}\}$	36	144	576	2304

 $r = 4$ 

weight	3	4	5
$\partial_n$	16	92	476
$\partial_n^{(c)}$	16	92	488
lin.	16	92	488
$\#\{\text{index set}\}$	80	400	2000

 $r = 5$ 

weight	3	4	5
$\partial_n$	25	170	1045
$\partial_n^{(c)}$	25	170	1065
lin.	25	170	1065
$\#\{\text{index set}\}$	150	900	5400

 $r = 6$ 

weight	3	4
$\partial_n$	36	282
$\partial_n^{(c)}$	36	282
lin.	36	282
$\#\{\text{index set}\}$	252	1764

## REFERENCES

- [1] T. Arakawa and M. Kaneko, *On multiple  $L$ -values*, J. Math. Soc. Japan 56 (2004), no. 4, 967–991.
- [2] A. Connes and H. Moscovici, *Hopf algebras, cyclic cohomology and the transverse index theorem*, Comm. Math. Phys. 198 (1998), no. 1, 199–246.
- [3] P. Deligne and A. Goncharov, *Groupes fondamentaux motiviques de Tate mixte*, Ann. Sci. Ecole Norm. Sup. (4) 38 (2005), no. 1, 1–56.
- [4] L. Euler, *Demonstratio insignis theorematis numerici circa uncias potestatum binomialium*, Nova Acta Academiae Scientiarum Imperialis Petropolitinae 15, (1799/1802), pp. 33–43.
- [5] A. Gel'fond, *Calculus of Finite Differences*, Hindustan, 1971.
- [6] A. Goncharov, *Multiple  $\zeta$ -values, Galois groups, and geometry of modular varieties*, European Congress of Mathematics, Vol. I (Barcelona, 2000), 361–392, Progr. Math., 201, Birkhauser, Basel, 2001.
- [7] M. Hoffman, *The algebra of multiple harmonic series*, J. Algebra 194 (1997), 477–495.
- [8] K. Ihara, M. Kaneko and D. Zagier : *Derivation and double shuffle relations for multiple zeta values*, Compos. Math. 142-02 (2006), 307–338.
- [9] S. Izumi, *Teisa-hô*, Iwanami, 1934 (in Japanese).
- [10] M. Kaneko, *On an extension of the derivation relation for multiple zeta values*, The Conference on  $L$ -Functions, edited by L. Weng and M. Kaneko, World Scientific (2007), 89–94.
- [11] G. Kawashima, *A class of relations among multiple zeta values*, submitted.
- [12] T. Tanaka, *On extended derivation relations for multiple zeta values*, submitted.
- [13] S. Wakiyama, *On the commutativity of twisted derivations for multiple  $L$ -values*, master's thesis (in Japanese), Kyushu University, 2007.

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